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FULL PAPER

Molecular descriptors of some graph operations through m-polynomial

Raad Sehen Haoer

[a]*Department of Mathematics, Open Educational College, Ministry of Education, Al-Qadisiya Centre, Iraq*

The most general algebraic polynomial to obtain a massive number of degree-based topological indices for a specific family of structures is the M-polynomial. In this paper, the M-polynomial of some graph operations was derived including join, corona product, strong product, tensor product, splice, and link of regular graphs. By using those expressions, numerous degree-based indices of the aforesaid operations were computed. The explicit expressions of the indices were also derived for some particular graphs.

***Corresponding Author:**
Raad Sehen Haoer
Email: raadsehen@gmail.com
Tel.: +9647811873034

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Introduction

The graph theory includes an important method called the topological index (or molecular descriptor) for correlating the physiochemical activity of molecular and networks. A molecular-graph is a simple connected graph in which nodes-edges are assumed to be atoms-chemical bonds of a compound. Topological index is nothing but a number obtained from molecular graphs which describes the molecular graph topology and is an invariant for isomorphic graphs. Such descriptors are considered to be fantastic tools in QSPR and QSAR modelling. Harold Wiener introduced the idea of topological index when he worked on boiling point alkanes in 1947 [1]. Several algebraic polynomials are spotted in the literary history to resolve the painstaking strategy of computing indices utilizing their customary understandings of a particular class of graphs [2,3,4,34-42]. The M-polynomial [15] plays an important role in computing a huge proportion of indices of a specific group of graphs in the scenario of degree-based topological-indices. If $V(G)$ and $E(G)$ represent the node and edge collection of a graph G, respectively, then degree of $v \in V(G)$, written as d_v, is defined as the number of members in $E(G)$ incident on v. Let n_i and m_i be the number of elements in $V(Gi)$, $E(Gi)$, respectively. If degreeof all nodes of G is r, then it is called the r-regular graph. The join G_1, G_2 is generated by connecting each node belongs to G_1 to each node of G_2. Corona product [7] of G_1 and G_2 is formed by considering one copy of G_1 and n_1 copy of G_2 and connecting the i-th node of G_1 to all nodes belonging to i-th part of G_2. The strong product [8] of *two* graphs contains node set $V(G1) \times V(G2)$ and (u_1, v_1) is adjacent with (u_2, v_2) iff $u_1 = u_2$ with $v_1 \sim v_2$ or $v_1 = v_2$ and $u_1 \sim u_2$. Tensor product contains the same node collection and (u_1, v_1) is adjacent with (u_2, v_2) if $u_1 \sim u_2$ and $v_1 \sim v_2$ [9,10,11,14]). Splice [12,13] of these two graphs is constructed by identifying the nodes v_1, v_2 in $G_1 \cup G_2$. The link [14] of these two graphs is constructed by joining the nodes v_1 and v_2 with an edge in the unionof G_1 and G_2.

Basavanagoud *et al.* [16] computed explicit expressions of algebraic polynomials for numerous operations. Mondal *et al.* [5] derived neighborhood Zagreb index of some graph operations. The present author obtained indices of various networks and dendrimer structures [17,18,19,20]. Khalifeh *et al.* [21] obtained Zagreb indices of different graph operations. The intention of the current report is to compute different degree-based topological descriptors of graph operations of regular graphs *via* M-polynomial.

TABLE 1 Relation between degree-based indices and M-polynomial

Topological Index	f(x,y)	Derivation from $M(G;x,y)$
First Zagreb Index	$x+y$	$(D_x+D_y)(M(G;x,y))\|_{x=y=1}$
Second Zagreb Index	xy	$(D_xD_y)(M(G;x,y))\|_{x=y=1}$
Modified Second Zagreb Index	$1/xy$	$(S_xS_y)(M(G;x,y))\|_{x=y=1}$
Symmetric Division Index	x^2+y^2/xy	$(D_xS_y+S_xD_y)(M(G;x,y))\|_{x=y=1}$
Harmonic Index	$2/x+y$	$2S_xJ(M(G;x,y))\|_{x=1}$
Inverse sum Index	$xy/x+y$	$S_xJD_xD_y(M(G;x,y))\|_{x=1}$
General sum connectivity	$(x+y)^{\alpha}$	$D_x^{\alpha}J(M(G;x,y)))\|_{x=1}$
First general Zagreb	$x^{\alpha-1}+y^{\alpha-1}$	$(D_x^{a-1}+D_x^{a-1})(M(G;x,y))\|_{x=1}$

Preliminaries

Lemma 2. If G is r-regular with n nodes and m edges, then the total number of edges is $|E|={rn}/{2}$.

Definition 1. The formulation of M-polynomial of G is defined as follows:

$$M_t(G;tx,y)=\sum_{p\leq q} m_{pq}(G)x^p y^q$$

Where, $m_{pq}(G)$ is the total count of connections u~v for which $\{d_u, d_v\}=\{p, q\}$. Gutman and Trinajstić presented Zagreb indices [23]. The first Zagreb-index is formulated as follows:

$$M_1(G)=\sum_{v\in V(G)}(d_v)^2$$

The second Zagreb descriptor is formed as:

$$M_2(G)=\sum_{uv\in E(G)} d_u\, d_v.$$

For more details about this indices, see [5,6,22,32,33,21].

The second modified Zagreb index is defined as follows:

$$^{m}M_2(G)=\sum_{uv\in E(G)}\frac{1}{d_ud_v}$$

Bollobas and Erdos [24] and Amic *et al.* [25] initiated the concept of generalized Randić index and illustrated in a broad range in mathematical chemistry [26]. For more details, see [28,27]. Such index is formulated as follows:

$$R_{\alpha}(G)=\sum_{uv\in E(G)}(d_ud_v)^{\alpha}$$

The inverse Randić index is formulated as:

$$RR_{\alpha}(G)=\sum_{uv\in E(G)}\frac{1}{(d_ud_v)^{\alpha}}$$

Symmetric division index is formulated as:

$$SDD(G)=\sum_{uv\in E(G)}\{\frac{min(d_u,d_v)}{max(d_u,d_v)}+\frac{max(d_u,d_v)}{min(d_u,d_v)}\}.$$

Harmonic index [29] is formulated as:

$$H(G)t = \sum_{uv\in E(G)} \frac{2}{d_u + d_v}.$$

The inverse sum index [30] is formulated as:

$$I(G)t = \sum_{uv\in E(G)} \frac{d_u d_v}{d_u + d_v}.$$

The augmented Zagreb index [31] is formulated as follows:

$$A(G) = \sum_{uv\in E(G)} \{\frac{d_u d_v}{d_u + d_v - 2}\}^3.$$

The way by which indices based on degree are recovered from Mt-polynomial is presented in Table 1.

Main results

In this section, we discuss the M-polynomial of different graph operations namely join, corona product, strong product, tensor product, splice, link, and explore the topological indices for those operations. We utilize the following lemma to obtain the main results.

Lemma 3. [16] For a a r-regular graph with n vertices and m edge connections, we have $M(G;tx,y)=mx^r y^r$.

Where t is

$$D_x(f(x,y))t = tx\frac{\partial(f(x,y))}{\partial x},$$

$$D_y(f(x,y))t = ty\frac{\partial(f(x,y))}{\partial y},$$

$$S_x(f(x,y)) = \int_0^x \frac{tf(t,y)t}{ttt}dt.$$

Join

We compute the *M*-polynomial of join G_1+G_2 in the following theorem.

Theorem 1. *If G_1,tG_2 are $r_{1,t}r_2$-regular, respectively, then for n_1-$n_2\geq r_1$-r_2, we find*

$$M(G_1+G_2) = n_1n_2x^{r_1+n_2}y^{r_2+n_1} + m_1x^{r_{1t}+tn_2}\, y^{r_1+n_2} + m_2x^{r_2+n_1}y^{r_2+n_1}.$$

Proof. Edge partition of G_1+G_2 is as follows:

$|E_{\{r_1+n2,r2+n1\}}|=|\{uv\in E(G_1+G_2):d_u=r_1+n_2,d_v=r_2+n_1|$ $=n_1n_2$.

$|E_{\{r_1+n2,r1+n2\}}|=|\{uv\in E(G_1+G_2):d_u=r_{1t}+n_2,d_v$ $=r_{1t}+n_2\}|$ $=m_1$.

$|E_{\{r_2+n1,r2+n1\}}|=|\{uv\in E(G_1+G_2):d_u=r_2+n_1,d_v=r_2+n_1|$ $=m_2$.

The *Mt*-polynomial of *Gt* is derived as follows

$$Mt(G_1+G_2) = \sum_{ti\leqslant jt} m_{ij}\,(G_1+G_2)x^{it}y^{jt}$$

$$= \sum_{r_1+n_2\leqslant r_2+n_1} m_{ij}\,(G_1+G_2)x^{r_1+n_2}y^{r_2+n_1} + \sum_{r_1+n_2\leqslant r_1+n_2} m_{ij}\,(G_1+G_2)x^{r_1+n_2}y^{r_1+n_2} + \sum_{r_2+n_1\leqslant r_2+n_1} m_{ij}\,(G_1+G_2)x^{r_2+n_1}y^{r_2+n_1}$$

$$= |E_{\{r_1+n_2,r_2+n_1\}}|x^{r_1+n_2}y^{r_2+n_1} + |E_{\{r_1+n_2,r_1+n_2\}}|x^{r_1+n_2}y^{r_1+n_2} + |E_{\{r_2+n_1,r_2+n_1\}}|x^{r_2+n_1}y^{r_2+n_1}$$

$$= n_1n_2x^{r_1+n_2}y^{r_2+n_1} + m_1x^{r_1+n_2}y^{r_1+n_2} + m_2x^{r_2+n_1}y^{r_2+n_1}.$$

Now employing such expression of *Mt*-polynomial, one easily finds degree-based topological index for G_1+G_2 in the following theorem.

Theorem 2. If G_1+G_2 be the join of graphs G_1 and G_2. Then, we have:

1. $M_1(G_1+G_2)=n_1n_2(r_{1t}+tr_{2t}+tn_{1t}+tn_2)+2m_1(r_{1t}+tn_2)+2m_2(r_{2t}+tn_1)$.

2. $M_2(G_1+G_2)=n_1n_2(r_2+n_1)(r_1+n_2)+m_1(r_1+n_2)^2+m_2(r_2+n_1)^2$.

3. $M_2^m(G_1+G_2) = \frac{n_1n_2}{(r_1+n_2)t(r_2+n_1)} + \frac{m_1}{(r_1+n_2)^{2t}} + \frac{m_2}{(r_2+n_1)^2}.$

4. $S_D(G_1+G_2) = n_1n_2(\frac{tr_1+n_2}{tr_2+n_1} + \frac{r_2+n_{1t}}{r_1+n_{2t}}) + 2m_1 + 2m_2.$

5. $H(G_1+G_2) = \frac{2n_1n_2}{r_1+r_2+n_1+n_{2t}}t + \frac{m_1}{tr_1+n_{2t}} + \frac{m_2}{tr_2+n_{1t}}.$

6. $I_n(G_1+G_2)=\frac{n_1n_2(r_1+n_2)t(r_2+n_1)t}{r_1+r_2+n_1+n_2}+\frac{tm_{1t}(r_1+n_2)t}{2}+\frac{tm_{2t}(r_2+n_1)}{2}$.

7. $\chi_\alpha(G_1+G_2)=n_1n_2(r_{1t}+tr_{2t}+tn_{1t}+tn_2)^{\alpha+}2^\alpha m_1(r_{1t}+tn_2)^\alpha+2^\alpha m_2(r_{2t}+tn_1)^\alpha$.

8. $M^\alpha(G_1+G_2)=n_1n_2(r_1+n_2)^{\alpha-1}+n_1n_2(r_2+n_1)^{\alpha-1}+2m_1(r_{1t}+tn_2)^{\alpha-1}+2m_2(r_{2t}+tn_1)^{\alpha-1}$.

Proof. Consider t $M((G_1+G_2);x,y)$t= t $f(x,y)=n_1n_2x^{r_1+n_2}y^{r_2+n_1}+m_1x^{r_1+n_2}y^{r_1+n_2}+m_2x^{r_2+n_1}y^{r_2+n_1}$.

Then, we have:

$D_x(f(x,y))=n_1n_2(r_1+n_2)x^{r_1+n_2}y^{r_2+n_1}+m_1(r_1+n_2)x^{r_{1t}+tn_2}y^{r_1+n_2}+m_2(r_2+n_1)x^{r_2+n_1}y^{r_2+n_1}$.

$D_y(f(x,y))=n_1n_2(r_2+n_1)x^{r_1+n_2}y^{r_2+n_1}+m_1(r_1+n_2)x^{r_{1t}+tn_2}y^{r_1+n_2}+m_2(r_2+n_1)x^{r_2+n_1}y^{r_2+n_1}$.

$D_xD_y(f(x,y))=n_1n_2(r_1+n_2)(r_2+n_1)x^{r_1+n_2}y^{r_2+n_1}+m_1(r_1+n_2)(r_1+n_2)x^{r_{1t}+tn_2}y^{r_1+n_2}+m_2(r_2+n_1)(r_2+n_1)x^{r_2+n_1}y^{r_2+n_1}$.

$D_xD_y(f(x,y))=n_1n_2(r_1+n_2)(r_2+n_1)x^{r_1+n_2}y^{r_2+n_1}+m_1(r_1+n_2)^2x^{r_1+n_2}y^{r_1+n_2}+m_2(r_2+n_1)^2x^{r_2+n_1}y^{r_2+n_1}$.

$S_x(f(x,y))=\frac{n_1n_2}{r_1+n_2}x^{(r_{1t}+tn_2)}y^{(r_{2t}+tn_1)}+\frac{m_1}{(r_{1t}+tn_2)}x^{(r_{1t}+tn_2)}y^{r_1+n_2}+\frac{m_2}{(r_{2t}+tn_1)}x^{(r_{2t}+tn_1)}y^{(r_{2t}+tn_1)}$.

$S_y(f(x,y))=\frac{n_1n_2}{(r_{2t}+tn_1)}x^{(r_{1t}+tn_2)}y^{(r_{2t}+tn_1)}+\frac{m_1}{(r_{1t}+tn_2)}x^{(r_{1t}+tn_2)}y^{r_1+n_2}+\frac{m_2}{(r_{2t}+tn_1)}x^{(r_{2t}+tn_1)}y^{(r_{2t}+tn_1)}$.

$S_xS_y(f(x,y))=\frac{n_1n_2}{(r_1+n_2)(r_2+n_1)}x^{(r_{1t}+tn_2)}y^{(r_{2t}+tn_1)}+\frac{m_1}{(r_1+n_2)^2}x^{(r_{1t}+tn_2)}y^{(r_{1t}+tn_2)}+\frac{m_2}{(r_2+n_1)^2}x^{(r_{2t}+tn_1)}y^{(r_{2t}+tn_1)}$.

$D_xS_y(f(x,y))=\frac{n_1n_2(r_1+n_2)}{r_2+n_1}x^{r_1+n_2}y^{(r_{2t}+tn_1)}+m_1x^{r_1+n_2}y^{r_1+n_2}+m_2x^{r_2+n_1}y^{r_2+n_1}$.

$D_yS_x(f(x,y))=\frac{n_1n_2(r_2+n_1)}{r_1+n_2}x^{r_1+n_2}y^{r_2+n_1}+m_1x^{r_1+n_2}y^{r_1+n_2}+m_2x^{r_2+n_1}y^{r_2+n_1}$.

$S_xJD_xD_y(f(x,y))=\frac{n_1n_2(r_1+n_2)(r_2+n_1)}{tr_1+r_2+n_1+n_2}x^{r_1+r_2+n_1+n_2}+\frac{m_1(r_{1t}+tn_2)}{2}x^{2(r_1+n_2)}+\frac{m_2(r_{2t}+tn_1)}{2}x^{2(r_2+n_1)}$.

$D_x^\alpha J(f(x,y))=n_1n_2(r_1+r_2+n_1+n_2)^\alpha x^{r_1+r_2+n_1+n_2}+2^\alpha m_1(r_1+n_2)^\alpha x^{r_1+r_2+n_1+n_2}+2^\alpha m_2(r_2+n_1)^\alpha x^{r_1+r_2+n_1+n_2}$.

$(D_x^{\alpha-1}+D_y^{\alpha-1})(f(x,y))=n_1n_2(r_1+n_2)^{\alpha-1}+n_1n_2(r_2+n_1)^{\alpha-1}x^{r_1+n_2}y^{r_2+n_1}+2m_1(r_1+n_2)^{\alpha-1}x^{r_1+n_2}y^{r_1+n_2}+2m_2(r_2+n_1)^{\alpha-1}x^{r_2+n_1}y^{r_2+n_1}$.

$2S_xJ(f(x,y))=\frac{2n_1n_2}{r_{1t}+tr_{2t}+tn_{1t}+tn_2}x^{r_1+r_2+n_1+n_2}+\frac{m_1}{r_1+n_2}x^{2(r_1+n_2)}+\frac{m_2}{r_{2t}+tn_1}x^{2(r_{2t}+tn_1)}$.

By using Table 1, we have:

1. $tM_{1t}(G_1+G_2)=(D_{xt}+tD_y)t(f(x,y))t|_{x=y=1}=n_1n_2(r_{1t}+tr_{2t}+tn_{1t}+tn_2)+2m_1(r_1+n_2)+2m_2(r_{2t}+tn_1)$.
2. $tM_2t(G_1+G_2)=(D_{xt}D_y)t(f(x,y))t|_{x=y=1}=n_1n_2(r_2+n_1)(r_1+n_2)+m_1(r_1+n_2)^2+m_2(r_2+n_1)^2$.

3. $M_2^m(G_1+G_2) = (S_{xt}S_y)t(f(x,y))t|_{x=y=1}$
$= \frac{n_1n_2}{(r_1+n_2)(r_2+n_1)} + \frac{m_1}{(r_1+n_2)^2} + \frac{m_2}{(r_2+n_1)^2}.$

4. $S_D(G_1+G_2) = (D_{xt}S_{yt} + tS_{xt}D_y)(f(x,y))t|_{x=y=1}$
$= n_1n_{2t}\left(\frac{r_{1t}+n_{2t}}{r_{2t}+tn_1} + \frac{r_{2t}+tn_1}{r_{1t}+n_{2t}}\right) + 2m_1 + 2m_2.$

5. $H(G_1+G_2) = 2(S_x)tJt(f(x,y))t|_{x=1}$
$= \frac{2n_1n_2}{r_{1t}+tr_{2t}+tn_{1t}+tn_2} + \frac{m_1}{r_{1t}+tn_2} + \frac{m_2}{r_{2t}+tn_1}.$

6. $I_n(G_1+G_2) = S_xJD_xD_y(f(x,y))|_{x=1}$
$= \frac{n_1n_2(r_{1t}+n_{2t})(r_{2t}+tn_1)}{r_{1t}+tr_{2t}+tn_{1t}+tn_2} + \frac{m_1(r_{1t}+tn_2)}{2} + \frac{m_2(r_2+n_1)}{2}.$

7. $\chi_\alpha(G_1+G_2) = D_x^\alpha(f(x,y))|_{x=1}$
$= n_1n_2(r_{1t}+tr_{2t}+tn_{1t}+tn_2)^\alpha + 2^\alpha m_1(r_{1t}+n_{2t})^\alpha + 2^\alpha m_2(r_2+n_1)^\alpha.$

8. $M_1^\alpha(G_1+G_2) = (D_{xt}^{\alpha-1} + D_{yt}^{\alpha-1})t(f(x,y))|_{x=y=1}$
$= n_1n_2(r_1+n_2)^{\alpha-1} + n_1n_2(r_2+n_1)^{\alpha-1}$
$+2m_1(r_1+n_2)^{\alpha-1} + 2m_2(r_2+n_1)^{\alpha-1}.$

This completes the proof.■

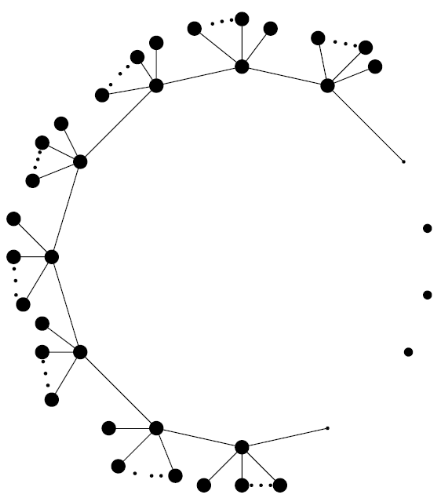

FIGURE 1 The t-throny graph of C_n

The join of $\bar{K}_n$ and $\bar{K}_t$ yields the complete bipartite graph $K_{n,t}$. Now by using Theorem 1, we obtain the following corollary.

Corollary 1. The M-polynomial of complete bipartite graph $K_{n,t}$ is given by $M(K_{n,t})=ntx^ty^n$.

By using Theorem 4, we obtain the following corollary.

Corollary 2. *The topological indices of $K_{n,t}$ are given by,*

1. $M_1(K_{n,t})=nt(n+t)$,
2. $M_2(K_{n,t})=n^2t^2$,
3. $M^m(K_{n,t})=1$,
4. $S_D(K_{n,t})=n^2+t^2$,
5. $H(K_{n,t}) = \frac{2nt}{n+t}$,
6. $I_n(K_{n,t}) = \frac{n^2t^2}{n+t}$,
7. $\chi_\alpha(K_{n,t})=nt(n+t)^\alpha$.
8. $M^\alpha(K_{n,t})=nt(n^{\alpha-1}+t^{\alpha-1})$.

Corona product

We compute the *M*-polynomial of the corona product of G_1 and G_2 in the following theorem.

Theorem 3. If G_1, tG_2 are $r_{1,t}r_2$-regular, respectively, then for $r_1+n_2-r_2 \geq 1$, we find:
$M(G_1 \odot G_2) = n_1n_2x^{r_2+1}y^{r_1+n_2}$
$+m_2n_1x^{r_2+1}y^{r_2+1} + m_1x^{r_1+n_2}y^{r_1+n_2}.$

Proof. The edge set of $G_1 \odot G_2$ has partitions as follows,

$|E_{\{r_2+1,tr_1+n_2\}}|=|\{uv\in tE(G_1\odot G_2): d_u=r_2+1, d_v=r_1+n_2\}|=n_1n_2.$

$|E_{\{r_2+1,tr_2+1\}}|=|\{uv\in E(G_1\odot G_2): d_u=r_2+1, d_v=tr_2+1\}|=m_2n_1.$

$|E_{\{r_1+n_2,tr_{1t}+tn_2\}}|=|\{uv\in E(G_1\odot G_2): d_u=r_1+n_2, d_v=r_1+n_2\}|=m_1.$

Now we clearly have

$$M(G) = \sum_{it\leqslant tj} m_{ij}(G)tx^{it}y^{jt}$$
$$= \sum_{r_2+1\leqslant r_1+n_2} m_{ij}(G_1 \odot G_2)x^{r_2+1}y^{r_1+n_2}$$

$$+\sum_{r_2+1\leqslant r_2+1} m_{ij}(G_1 \odot G_2)x^{r_2+1}y^{r_2+1}$$
$$+\sum_{r_1+n_{2t}\leqslant tr_1+n_2} m_{ij}(G_1 \odot G_2)x^{r_1+n_2}y^{r_1+n_2}$$
$$= |E_{\{tr_{2t}+1,tr_{1t}+n_{2t}\}}|x^{r_2+1}y^{r_1+n_{2t}} + |E_{\{r_2+1,tr_2+1\}}|x^{r_2+1}y^{r_2+1}$$
$$+|E_{\{tr_{1t}+tn_2,tr_{1t}+tn_{2t}\}}|x^{r_1+n_2}y^{r_1+n_2}$$
$$= n_1n_2x^{r_{2t}+1}y^{r_{1t}+tn_2} + m_2n_1x^{r_{2t}+1}y^{r_{2t}+1} + m_1x^{r_{1t}+tn_2}y^{r_{1t}+tn_2}.$$

This completes the proof.■

Theorem 4. If $G_1 \odot G_2$ be the Corona product of the two graphs G_1 and G_2. Then, we have:

1. $M_1(G_1 \odot G_2) = n_1n_2(r_{1t} + tr_2 + tn_{2t} + 1) + 2m_2n_1(r_{2t} + t1) + 2m_1(r_{1t} + tn_2).$
2. $M_2(G_1 \odot G_2) = n_1n_2(r_{2t} + t1)(r_{1t} + tn_2) + m_2n_1(r_{2t} + t1)^2 + m_1(r_{1t} + tn_2)^2$
3. $M_2^m(G_1 \odot G_2) = \frac{n_1n_2}{(r_{1t} + tn_2)(r_{2t} + t1)} + \frac{m_2n_1}{(r_{2t} + t1)^2} + \frac{m_1}{(r_{1t} + tn_2)^2}$
4. $S_D(G_1 \odot G_2) = n_1n_2(\frac{r_2 + 1}{r_{1t} + tn_2} + \frac{r_1 + n_2}{r_2 + 1}) + 2m_2n_1 + 2m_1.$
5. $H(G_1 \odot G_2) = \frac{2n_1n_2}{r_{1t} + tr_2 + tn_{2t} + 1} + \frac{m_2n_1}{r_{2t} + t1} + \frac{m_1}{r_{1t} + tn_2}$
6. $I_n(G_1 \odot G_2) = \frac{n_1n_2(r_{1t} + tn_2)(r_{2t} + t1)}{r_1 + r_2 + n_1 + 1} + \frac{m_2n_1(r_{2t} + t1)}{2} + \frac{m_1(r_{1t} + tn_2)}{2}.$
7. $\chi_\alpha(G_1 \odot G_2) = n_1n_2(r_1 + r_2 + n_2 + 1)^\alpha + 2^\alpha m_2n_1(r_1 + n_2)^\alpha + 2^\alpha m_1(r_1 + n_2)^\alpha.$
8. $M_1^\alpha(G_1 \odot G_2) = n_1n_2(r_2 + 1)^{\alpha-1} + n_1n_2(r_{1t} + tn_2)^{\alpha-1} + 2m_2n_1(r_{2t} + 1)^{\alpha-1} + 2m_1(r_{1t} + tn_2)^{\alpha-1}.$

Proof. Lett$M((G_1 \odot G_2);x,y) = n_1n_2x^{r_2+1}y^{r_1+n_2} + m_2n_1x^{r_2+1}y^{r_2+1} + m_1x^{r_1+n_2}y^{r_1+n_2}$ *and*

$$f(x,y) = M(G_1 \odot G_2).$$

$$D_x(f(x,y)) = n_1n_2(r_2 + 1)x^{r_2+1}y^{r_1+n_2} + m_2n_1(r_2 + 1)x^{r_2+1}y^{r_2+1} + m_1(r_1 + n_2)x^{r_1+n_2}y^{r_1+n_2}.$$

$$D_y(f(x,y)) = n_1n_2(r_1 + n_2)x^{r_2+1}y^{r_1+n_2} + m_2n_1(r_2 + 1)x^{r_2+1}y^{r_2+1} + m_1(r_1 + n_2)x^{r_1+n_2}y^{r_1+n_2}.$$

$$D_xD_y(f(x,y)) = n_1n_2(r_2 + 1)(r_1 + n_2)x^{r_2+1}y^{r_1+n_2} + m_2n_1(r_2 + 1)^2x^{r_2+1}y^{r_2+1} + m_1(r_1 + n_2)^2x^{r_1+n_2}y^{r_1+n_2}.$$

$$S_x(f(x,y)) = \frac{n_1n_2}{r_2 + 1}x^{r_{2t}+t1}ty^{r_{1t}+tn_2} + \frac{m_2n_1}{r_{2t} + t1}x^{r_{2t}+t1}y^{r_{2t}+t1} + \frac{m_1}{r_1 + n_2}x^{r_{1t}+tn_2}y^{r_{1t}+tn_2}.$$

$$S_y(f(x,y)) = \frac{n_1n_2}{r_1 + n_2}x^{r_{2t}+t1}y^{r_{1t}+tn_2} + \frac{m_2n_1}{r_{2t} + t1}x^{r_{2t}+t1}y^{r_{2t}+t1} + \frac{m_1}{r_1 + n_2}x^{r_{1t}+tn_2}y^{r_{1t}+tn_2}.$$

$$S_xtS_{yt}(f(x,y)) = \frac{n_1n_2}{(r_{2t} + t1)(r_{1t} + tn_2)}x^{r_{2t}+t1}y^{r_{1t}+tn_2} + \frac{m_2n_1}{(r_2 + 1)^2}x^{r_2+1}y^{r_2+1} + \frac{m_1}{(r_1 + n_2)^2}x^{r_1+n_2}y^{r_1+n_2}.$$

$$D_xS_y(f(x,y)) = \frac{(n_1n_2)(r_2 + 1)}{r_1 + n_2}x^{r_2+1}y^{r_1+n_2} + m_2n_1x^{r_2+1}y^{r_2+1} + m_1x^{r_1+n_2}y^{r_1+n_2}.$$

$$S_xD_y(f(x,y)) = \frac{(n_1n_2)(r_1 + n_2)}{r_2 + 1}x^{r_2+1}y^{r_1+n_2} + m_2n_1x^{r_{2t}+t1}y^{r_{2t}+t1} + m_1x^{r_1+n_2}y^{r_{1t}+tn_2}.$$

$$2S_xJ(f(x,y)) = \frac{2n_1n_2}{r_{1t} + tr_{2t} + tn_{2t} + t1}x^{r_1+r_2+n_2+1} + \frac{m_2n_1}{r_2 + 1}x^{2(r_{2t}+t1)} + \frac{m_1}{r_{1t} + tn_2}x^{2(r_{1t}+tn_2)}.$$

$$S_xJD_xD_y(f(x,y)) = \frac{n_1n_2(r_1+n_2)(r_2+1)}{r_{1t}+tr_{2t}+tn_{2t}+t1}x^{r_{1t}+tr_{2t}+tn_{2t}+t1} + \frac{m_2n_1(r_2+1)}{2}x^{2(r_2+1)} + \frac{m_1(r_1+n_2)}{2}x^{2(r_1+n_2)}.$$

$$D_x^{\alpha}J(f(x,y)) = n_1n_2(r_{1t}+tr_{2t}+tn_{2t}+t1)^{\alpha}x^{r_{1t}+tr_{2t}+tn_{2t}+t1} + 2^{\alpha}m_2n_1(r_1x^{r_{1t}+tr_{2t}+tn_{2t}+t1}+n_2)^{\alpha} + 2^{\alpha}m_1(r_{1t}+tn_2)^{\alpha}x^{r_1+r_2+n_2+1}.$$

$$(D_{xt}^{\alpha-1}+D_{yt}^{\alpha-1})t(f(x,y)) = n_1n_2(r_{2t}+t1)^{\alpha-1} + n_1n_2(r_1+n_2)^{\alpha-1}x^{r_2+1}y^{r_1+n_2} + 2m_2n_1(r_2+1)^{\alpha-1}x^{r_2+1}y^{r_2+1} + 2m_1(r_1+n_2)^{\alpha-1}x^{r_1+n_2}y^{r_1+n_2}.$$

Using Table 1.we have,

1. $tM_{1t}(G_1 \odot G_2) = (D_{xt}+tD_y)t(f(x,y))t|_{x=y=1} = n_1n_2(r_{1t}+tr_{2t}+tn_{2t}+t1) + 2m_2n_1(r_{2t}+t1) + 2m_1(r_1+n_2).$

2. $tM_{2t}(G_1 \odot G_2) = (D_{xt}D_y)t(f(x,y))t|_{x=y=1} = n_1n_2(r_{2t}+t1)(r_1+n_2) + m_2n_1(r_{2t}+t1)^2 + m_1(r_{1t}+tn_2)^2$

3. $M_2^m(G_1 \odot G_2) = (S_{xt}S_y)t(f(x,y))t|_{x=y=1} = \frac{n_1n_2}{(r_1+n_2)(r_2+1)} + \frac{m_2n_1}{(r_2+1)^2} + \frac{m_1}{(r_1+n_2)^2}.$

4. $S_D(G_1 \odot G_2) = (D_{xt}S_y+S_{xt}D_y)t(f(x,y))t|_{x=y=1} = n_1n_2(\frac{r_2+1}{r_1+n_2}+\frac{r_1+n_2}{r_2+1}) + 2m_2n_1 + 2m_1.$

5. $H(G_1 \odot G_2) = 2(S_x)tJt(f(x,y))t|_{x=1} = \frac{2n_1n_2}{r_{1t}+tr_{2t}+tn_{2t}+t1} + \frac{m_2n_1}{r_{2t}+t1} + \frac{m_1}{r_{1t}+tn_2}.$

6. $I_n(G_1 \odot G_2) = S_xJD_xD_y(f(x,y))|_{x=1} = \frac{n_1n_2(r_{1t}+tn_2)(r_{2t}+t1)}{r_1+r_2+n_1+1} + \frac{m_2n_1(r_{2t}+t1)}{2} + \frac{m_1(r_{1t}+tn_2)}{2}.$

7. $\chi_{\alpha}(G_1 \odot G_2) = D_x^{\alpha}(f(x,y))|_{x=1} = n_1n_2(r_1+r_2+n_2+1)^{\alpha} + 2^{\alpha}m_2n_1(r_1+n_2)^{\alpha} + 2^{\alpha}m_1(r_1+n_2)^{\alpha}.$

8. $M_1^{\alpha}(G_1 \odot G_2) = (D_x^{\alpha-1}+D_y^{\alpha-1})(f(x,y))|_{x=y=1} = n_1n_2(r_{2t}+t1)^{\alpha-1} + n_1n_2(r_{1t}+tn_2)^{\alpha-1} + 2m_2n_1(r_{2t}+t1)^{\alpha-1} + 2m_1(r_{1t}+tn_2)^{\alpha-1}.$

This completes the proof.

The t-fold bristled graph $Brs_t(G)$ of G is obtained by corona prod-uct of G, $\bar{K}_t$. Such type of graph is also called as t-throny graph. By using the theorem, one can obtain the Mt-polynomial of $Brs_t(C_n)$ in the corollary given below.

Corollary 3. The M-polynomial of t-throny graph of C_n is given by:
$M(Brs_t(C_n))= ntxy^{2+t}+ nx^{2+t}y^{2+t}$.
By using Theorem 4, we obtain the following corollary.

Corollary 4. *The topological indices of $Brs_t(C_n)$ are given by:*

1. $M_1(Brs_t(C_n))= nt^2+5nt+4n$,
2. $M_2(Brs_t(C_n))=2n(t+2)(t+1)$,
3. $M_2^m(Brs_t(C_n)) = \frac{nt}{t+2} + \frac{n}{(t+2)^2}.$
4. $S_D(Brs_t(C_n)) = nt(\frac{1}{t+2}+t+2)+2n.$
5. $H(Brs_t(C_n)) = \frac{2nt}{t+3} + \frac{n}{t+2}.$
6. $I_n(Brs_t(C_n)) = \frac{nt(t+2)}{n+3} + \frac{n(t+2)}{2}.$
7. $\chi_{\alpha}(Brs_t(C_n))=nt(t+3)^{\alpha}+2^{\alpha}n(t+2)^{\alpha}$,
8. $M^{\alpha}(Brs_t(C_n))=nt+nt(t+2)^{\alpha-1}+2n(t+2)^{\alpha-1}$.

Strong product

The strong product of two graphs G_1 and G_2 is denoted by $G_1 \boxtimes G_2$. We compute the *M*-polynomial of the strong product of G_1 and G_2 in the following theorem.

Theorem 5. *Let G_1 and G_2 be two regular graphs of degree r_1 and r_2, respectively.Then, we have:*

$$M(G_1 \boxtimes G_2) = \frac{(n_1n_2-1)n_1n_2}{2} x^{n_1n_2-1} y^{n_1n_2-1}.$$

Proof. The strong product of two regular graphs G_1 and G_2 with degree r_1 and r_2, respectively, is also a regular graph of degree n_1n_2-1 with n_1n_2 vertices. Consequently, the result follows from the Lemma 2 and Lemma 3.■

Now by using this *M*-polynomial, we calculate some degree based topological indices of the $G_1 \boxtimes G_2$ in the following theorem.

Theorem 6. *If $G_1 \boxtimes G_2$ be the strong product. Then,*

$$1. M_1(G_1 \boxtimes G_2) = n_1n_2(n_1n_2-1)^2.$$

$$2. M_2(G_1 \boxtimes G_2) = \frac{n_1n_1(n_1n_2-1)^3}{2}.$$

$$3. M_2^m(G_1 \boxtimes G_2) = \frac{n_1n_2}{2(n_1n_2-1)}.$$

$$4. S_D(G_1 \boxtimes G_2) = n_1n_2(n_1n_2-1).$$

$$5. H(G_1 \boxtimes G_2) = \frac{n_1n_2}{2}.$$

$$6. I_n(G_1 \boxtimes G_2) = \frac{(n_1n_2-1)^2 n_1n_2}{4}.$$

$$7. \chi_\alpha(G_1 \boxtimes G_2) = 2^{\alpha-1} n_1n_2(n_1n_2-1)^{\alpha+1}.$$

$$8. M_1^\alpha(G_1 \boxtimes G_2) = n_1n_2(n_1n_2-1)^\alpha.$$

Proof. Let

$$M((G_1 \boxtimes G_2); x, y) = f(x,y) = \frac{n_1n_2(n_1n_2-1)}{2} x^{n_1n_2-1} y^{n_1n_2-1}.$$

Then,

$$D_x(f(x,y)) = \frac{n_1n_2(n_1n_2-1)}{2}(n_1n_2-1)x^{n_1n_2-1}y^{n_1n_2-1}.$$

$$D_y(f(x,y)) = \frac{n_1n_2(n_1n_2-1)}{2}(n_1n_2-1)x^{n_1n_2-1}y^{n_1n_2-1}.$$

$$D_xD_y(f(x,y)) = \frac{n_1n_2(n_1n_2-1)}{2}(n_1n_2-1)^2 x^{n_1n_2-1}y^{n_1n_2-1}.$$

$$S_x(f(x,y)) = \frac{n_1n_2}{2} x^{n_1n_2-1}y^{n_1n_2-1}.$$

$$S_y(f(x,y)) = \frac{n_1n_2}{2} x^{n_1n_2-1}y^{n_1n_2-1}.$$

$$S_xS_y(f(x,y)) = \frac{n_1n_2}{2(n_1n_2-1)} x^{n_1n_2-1}y^{n_1n_2-1}.$$

$$D_xS_y(f(x,y)) = \frac{n_1n_2(n_1n_2-1)}{2} x^{n_1n_2-1}y^{n_1n_2-1}.$$

$$2S_xJ(f(x,y)) = \frac{n_1n_2}{2} x^{2(n_1n_2-1)}.$$

Then, we have:

$$D_x(f(x,y)) = \frac{n_1n_2(n_1n_2-1)^2}{2}(n_1n_2-1)x^{n_1n_2-1}y^{n_1n_2-1}.$$

$$D_y(f(x,y)) = \frac{n_1n_2(n_1n_2-1)^2}{2}(n_1n_2-1)x^{n_1n_2-1}y^{n_1n_2-1}.$$

$$D_xD_y(f(x,y)) = \frac{n_1n_2(n_1n_2-1)^3}{2} x^{n_1n_2-1}y^{n_1n_2-1}.$$

$$S_xJD_xD_y(f(x,y)) = \frac{n_1n_2(n_1n_2-1)^2}{4} x^{2(n_1n_2-1)}.$$

$$D_x^\alpha J(f(x,y)) = 2^{\alpha-1} n_1n_2(n_1n_2-1)^{\alpha+1} x^{2(n_1n_2-1)}$$

$$(D_x^{\alpha-1} + D_y^{\alpha-1})(f(x,y)) = 2n_1n_2(n_1n_2-1)^\alpha x^{n_1n_2-1}y^{n_1n_2-1}.$$

By using Table 1, we have,

$$1. M_1(G_1 \boxtimes G_2) = (D_x + D_y)(f(x,y))|_{x=y=1} = n_1n_2(n_1n_2-1)^2.$$

$$2. M_2(G_1 \boxtimes G_2) = (D_xD_y)(f(x,y))|_{x=y=1} = \frac{n_1n_1(n_1n_2-1)^3}{2}.$$

$$3. M_2^m(G_1 \boxtimes G_2) = (S_xS_y)(f(x,y))|_{x=y=1} = \frac{n_1n_2}{2(n_1n_2-1)}.$$

$$4. S_D(G_1 \boxtimes G_2) = (D_xS_y + S_xD_y)(f(x,y))|_{x=y=1} = n_1n_2(n_1n_2-1).$$

$$5. H(G_1 \boxtimes G_2) = 2(S_x)J(f(x,y))|_{x=1} = \frac{n_1n_2}{2}.$$

$6. I_n(G_1 \boxtimes G_2) = S_xJD_xD_y(f(x,y))|_{x=1} = \frac{(n_1n_2-1)^2 n_1n_2}{4}.$

$7. \chi_\alpha(G_1 \boxtimes G_2) = D_x^\alpha(f(x,y))|_{x=1} = 2^{\alpha-1}n_1n_2(n_1n_2-1)^{\alpha+1}.$

$8. M_1^\alpha(G_1 \boxtimes G_2) = (D_x^{\alpha-1} + D_y^{\alpha-1})(f(x,y))|_{x=y=1} = n_1n_2(n_1n_2-1)^\alpha.$

This completes the proof.■

Tensor product

The tensor product of two graphs G_1 and G_2 is denoted by $G_1 \times G_2$. We compute the *M*-polynomial of the tensor product of G_1 and G_2 in the following theorem.

Theorem 7. Let G_1 and G_2 be two regular graphs of degree r_1 and r_2, respectively. Then, we have:

$$M(G_1 \times G_2) = \frac{n_1n_2(n_1n_2-n_1-n_2+1)}{2} x^{n_1n_2-n_1-n_2+1} y^{n_1n_2-n_1-n_2+1}.$$

Proof. The tensor product of two regular graphs G_1 and G_2 with degree r_1 and r_2, respectively, is also a regular graph of degree n_1n_2-n_1-n_2+1 with n_1n_2 vertices. Consequently, the result follows from the Lemma 2 and Lemma 3.■

$$S_xD_y(f(x,y)) = \frac{n_1n_2(n_1n_2-1)}{2} x^{n_1n_2-1} y^{n_1n_2-1}$$

Theorem 8. The topological indices of $G_1 \times G_2$ are given by:

$1. M_1(G_1 \times G_2) = n_1n_2(n_1n_2-n_1-n_2+1)^2.$

$2. M_2(G_1 \times G_2) = \frac{n_1n_2(n_1n_2-n_1-n_2+1)^3}{2}.$

$3. M_2^m(G_1 \times G_2) = \frac{n_1n_2}{2(n_1n_2-n_1-n_2+1)}.$

$4. S_D(G_1 \times G_2) = n_1n_2(n_1n_2-n_1-n_2+1).$

$5. H(G_1 \times G_2) = \frac{n_1n_2}{2}.$

$6. I_n(G_1 \times G_2) = \frac{n_1n_2(n_1n_2-n_1-n_2+1)^2}{4}.$

$7. \chi_\alpha(G_1 \times G_2) = 2^{\alpha-1}n_1n_2(n_1n_2-n_1-n_2+1)^{\alpha+1}.$

$8. M_1^\alpha(G_1 \times G_2) = n_1n_2(n_1n_2-n_1-n_2+1)^\alpha.$

Proof. Let

$$M((G_1 \times G_2);x,y) = f(x,y) = \frac{n_1n_2(n_1n_2-n_1-n_2+1)}{2} x^{n_1n_2-1} y^{n_1n_2-1}.$$

Then, we have:

$$D_x(f(x,y)) = \frac{n_1n_2(n_1n_2-n_1-n_2+1)^2}{2} x^{n_1n_2-n_1-n_2+1} y^{n_1n_2-n_1-n_2+1}.$$

$$D_y(f(x,y)) = \frac{n_1n_2(n_1n_2-n_1-n_2+1)^2}{2} x^{n_1n_2-n_1-n_2+1} y^{n_1n_2-n_1-n_2+1}.$$

$$D_xD_y(f(x,y)) = \frac{n_1n_2(n_1n_2-n_1-n_2+1)^3}{2} x^{n_1n_2-n_1-n_2+1} y^{n_1n_2-n_1-n_2+1}.$$

$$S_x(f(x,y)) = \frac{n_1n_2}{2} x^{n_1n_2-n_1-n_2+1} y^{n_1n_2-n_1-n_2+1}.$$

$$S_y(f(x,y)) = \frac{n_1n_2}{2} x^{n_1n_2-n_1-n_2+1} y^{n_1n_2-n_1-n_2+1}.$$

$$S_xS_y(f(x,y)) = \frac{n_1n_2}{2(n_1n_2-n_1-n_2+1)} x^{n_1n_2-n_1-n_2+1} y^{n_1n_2-n_1-n_2+1}.$$

$$D_xS_y(f(x,y)) = \frac{n_1n_2(n_1n_2-n_1-n_2+1)}{2} x^{n_1n_2-n_1-n_2+1} y^{n_1n_2-n_1-n_2+1}.$$

$$S_xD_y(f(x,y)) = \frac{n_1n_2(n_1n_2-n_1-n_2+1)}{2} x^{n_1n_2-n_1-n_2+1} y^{n_1n_2-n_1-n_2+1}.$$

$$2S_xJ(f(x,y)) = \frac{n_1n_2}{2} x^{2(n_1n_2-n_1-n_2+1)}.$$

$$S_xJD_xD_y(f(x,y)) = \frac{n_1n_2(n_1n_2-n_1-n_2+1)^2}{4} x^{2(n_1n_2-n_1-n_2+1)}.$$

$$D_x^\alpha J(f(x,y)) = 2^{\alpha-1}n_1n_2(n_1n_2-n_1-n_2+1)^{\alpha+1} x^{2(n_1n_2-n_1-n_2+1)}.$$

$(D_x^{\alpha-1}+D_y^{\alpha-1})(f(x,y))$
$=n_1n_2(n_1n_2-n_1-n_2+1)^{\alpha}x^{n_1n_2-n_1-n_2+1}y^{n_1n_2-n_1-n_2+1}$.

Using Table 1, we have

$1. M_1(G_1\times G_2)=(D_x+D_y)(f(x,y))|_{x=y=1} = n_1n_2(n_1n_2-n_1-n_2+1)^2$

$2. M_2(G_1\times G_2)=(D_xD_y)(f(x,y))|_{x=y=1} = \frac{n_1n_2(n_1n_2-n_1-n_2+1)^3}{2}$.

$3. M_2^m(G_1\times G_2)=(S_xS_y)(f(x,y))|_{x=y=1} = \frac{n_1n_2}{2(n_1n_2-n_1-n_2+1)}$.

$4. S_D(G_1\times G_2)=(D_xS_y+S_xD_y)(f(x,y))|_{x=y=1} = n_1n_2(n_1n_2-n_1-n_2+1)$.

$5. H(G_1\times G_2)=2(S_x)J(f(x,y))|_{x=1}=\frac{n_1n_2}{2}$.

$6. I_n(G_1\times G_2)=S_xJD_xD_y(f(x,y))|_{x=1} = \frac{n_1n_2(n_1n_2-n_1-n_2+1)^2}{4}$.

$7. \chi_{\alpha}(G_1\times G_2)=D_x^{\alpha}(f(x,y))|_{x=1} = 2^{\alpha-1}n_1n_2(n_1n_2-n_1-n_2+1)^{\alpha+1}$.

$8. M_1^{\alpha}(G_1\times G_2)=(D_x^{\alpha-1}+D_y^{\alpha-1})(f(x,y))|_{x=y=1} = n_1n_2(n_1n_2-n_1-n_2+1)^{\alpha}$.

Splice

The splice of two graphs G_1 and G_2 is denoted by $G_1.G_2$. The M-polynomial of the splice of G_1 and G_2 is obtained in the following theorem.

Theorem 9. Let G_1 and G_2 be two regular graphs of degree r_1 and r_2, respectively. Then, we have:

$M(G_1.G_2)=(n_1-1)x^{r_1}y^{r_1+r_2}+(n_2-1)x^{r_2}y^{r_1+r_2}+(m_1-r_1)x^{r_1}y^{r_1}+(m_2-r_2)x^{r_2}y^{r_2}$.

Proof. The edge set of $G_1\cdot G_2$ has the following partitions.

$|E_{\{r_1,r_1+r_2\}}|=|\{uv\in E(G_1\cdot G_2): d_u=r_1, d_v=r_1+r_2\}|=n_1-1$.

$|E_{\{r_2,r_1+r_2\}}|=|\{uv\in E(G_1\cdot G_2): d_u=r_2, d_v=r_1+r_2\}|=n_2-1$.

$|E_{\{r_1,r_1\}}|=|\{uv\in E(G_1\cdot G_2): d_u=r_1, d_v=r_1\}|=m_1-r_1$.

$|E_{\{r_2,r_2\}}|=|\{uv\in E(G_1\cdot G_2): d_u=r_2, d_v=r_2\}|=m_2-r_2$.

Based on the definition, the M-polynomial of G is obtained as follows:

$$M(G)=\sum_{i\leqslant j} m_{ij}(G)x^iy^j$$
$$=\sum_{r_1\leqslant r_1+r_2} m_{ij}(G_1\cdot G_2)x^{r_1}y^{r_1+r_2}+\sum_{r_2\leqslant r_1+r_2} m_{ij}(G_1\cdot G_2)x^{r_2}y^{r_1+r_2}$$
$$+\sum_{r_1\leqslant r_1} m_{ij}(G_1\cdot G_2)x^{r_1}y^{r_1}+\sum_{r_2\leqslant r_2} m_{ij}(G_1\cdot G_2)x^{r_2}y^{r_2}$$
$$=|E_{\{r_1,r_1+r_2\}}|x^{r_1+1}y^{r_1+r_2}+|E_{\{r_2,r_1+r_2\}}|x^{r_2}y^{r_1+r_2}$$
$$+|E_{\{r_1,r_1\}}|x^{r_1}y^{r_1}+|E_{\{r_2,r_2\}}|x^{r_2}y^{r_2}$$
$$=(n_1-1)x^{r_1}y^{r_1+r_2}+(n_2-1)x^{r_2}y^{r_1+r_2}$$
$$+(m_1-r_1)x^{r_1}y^{r_1}+(m_2-r_2)x^{r_2}y^{r_2}.$$

This completes the proof.■

Now by using this M-polynomial, we calculate some degree based topological indices of the $G_1\cdot G_2$ in the following theorem.

Theorem 10. The degree based topological indices of $G_1\cdot G_2$ are given by:

$1. M_1(G_1\cdot G_2)=(n_1-1)(2r_1+r_2)+(n_2-1)(r_1+2r_2)+2r_1(m_1-r_1)+2r_2(m_2-r_2)$.

$2. M_2(G_1\cdot G_2)=(r_1+r_2)(r_1(n_1-1)+r_2(n_2-1))+r^2(m_1-r_1)+r^2(m_2-r_2)$.

$3. M_2(G_1\cdot G_2)=(r_1+r_2)(r_1(n_1-1)+r_2(n_2-1))+r_1^2(m_1-r_1)+r_2^2(m_2-r_2)$.

$4. M_2^m(G_1\cdot G_2)=\frac{n_1-1}{r_1(r_1+r_2)}+\frac{n_2-1}{r_2(r_1+r_2)}+\frac{m_1-r_1}{r_1^2}+\frac{m_2-r_2}{r_2^2}$.

$5. S_D(G_1\cdot G_2)=\frac{(n_1-1)r_1}{(r_1+r_2)}+\frac{(n_2-1)r_2}{r_1+r_2}+\frac{(n_1-1)(r_1+r_2)}{r_1}+\frac{(n_2-1)(r_1+r_2)}{r_2}+2(m_1-r_1)+2(m_2-r_2)$

$$6. H(G_1 \cdot G_2) = \frac{2(n_1-1)}{2r_1+r_2} + \frac{2(n_2-1)}{r_1+2r_2} + \frac{m_1-r_1}{r_1} + \frac{m_2-r_2}{r_2}$$

$$7. I_n(G_1 \cdot G_2) = \frac{r_1(r_1+r_2)(n_1-1)}{2r_1+r_2} + \frac{r_2(r_1+r_2)(n_2-1)}{r_1+2r_2} + \frac{r_1(m_1-r_1)}{2} + \frac{r_2(m_2-r_2)}{2}$$

8. $\chi_{\alpha}(G_1 \cdot G_2)= (n_1-1)(2r_1+r_2)^{\alpha}+(n_2-1)(r_1+2r_2)^{\alpha}+2^{\alpha}r^{\alpha}(m_1-r_1)+2^{\alpha}r_2(m_2-r_2)^{\alpha}$.

9. $M^{\alpha}(G_1 \cdot G_2)=(n_1-1)r_1^{\alpha-1}(n_2-1)r^{\alpha-1}+2(m_1-r_1)r^{\alpha-1}+2(m_2-r_2)r^{\alpha-1}+(r_1+r_2)^{\alpha-1}(n_1+n_2-2)$.

Link

The link of G_1 and G_2 is denoted by $G_1{\sim}G_2$. We compute the M-polynomialof the link of G_1 and G_2 in the following theorem.

Theorem 11. Let G_1 and G_2 be two regular graph of degree r_1 and r_2, respectively. Then, we have:

$M(G_1{\sim}G_2)= x^{r_2+1}y^{r_1+1}+(n_1-1)x^{r_1}y^{r_1+1}+(n_2-1)x^{r_2}y^{r_1+1}+(m_1-r_1)x^{r_1}y^{r_1}+(m_2-r_2)x^{r_2}y^{r_2}$.

Proof. The edge set of $G_1{\sim}G_2$ is partitioned as follows:

$|E_{\{r_2+1,r_1+1\}}|=|\{uv\in E(G_1{\sim}G_2): d_u=r_2+1, d_v=r_1+1\}|=1$.

$|E_{\{r_1,r_1+1\}}|=|\{uv\in E(G_1{\sim}G_2): d_u=r_1, d_v=r_1+1\}|=n_1-1$.

$|E_{\{r_2,r_2+1\}}|=|\{uv\in E(G_1{\sim}G_2): d_u=r_2, d_v=r_2+1\}|=n_2-1$.

$|E_{\{r_1,r_1\}}|=|\{uv\in E(G_1{\sim}G_2): d_u=r_1, d_v=r_1\}|=m_1-r_1$.

$\}|=|\{uv\in E(G_1{\sim}G_2): d_u=r_2, d_v=r_2\}|=m_2-r_2$.

From the definition, the M-polynomial of $G_1{\sim}G_2$ is obtained as follows:

$$M(G_1{\sim}G_2) = \sum_{i\leqslant j} m_{ij}(G_1{\sim}G_2)x^i y^j$$
$$= \sum_{r_2+1\leqslant r_1+1} m_{ij}(G_1{\sim}G_2)x^{r_2+1}y^{r_1+1} + \sum_{r_1\leqslant r_1+1} m_{ij}(G_1{\sim}G_2)x^{r_1}y^{r_1+1}$$
$$+ \sum_{r_2\leqslant r_2+1} m_{ij}(G_1{\sim}G_2)x^{r_2}y^{r_2+1} + \sum_{r_1\leqslant r_1} m_{ij}(G_1{\sim}G_2)x^{r_1}y^{r_1}$$
$$+ \sum_{r_2\leqslant r_2} m_{ij}(G_1{\sim}G_2)x^{r_2}y^{r_2}$$
$$= |E_{\{r_2+1,r_1+1\}}|x^{r_1+1}y^{r_1+r_2} + |E_{\{r_1,r_1+1\}}|x^{r_1+1}y^{r_1+r_2} + |E_{\{r_2,r_2+1\}}|x^{r_2}y^{r_1+r_2} + |E_{\{r_1,r_1\}}|x^{r_1}y^{r_1} + |E_{\{r_2,r_2\}}|x^{r_2}y^{r_2}$$
$$= x^{r_2+1}y^{r_1+1} + (n_1-1)x^{r_1}y^{r_1+1} + (n_2-1)x^{r_2}y^{r_1+1} + (m_1-r_1)x^{r_1}y^{r_1} + (m_2-r_2)x^{r_2}y^{r_2}.$$

This completes the proof.■

Theorem 12. *The topological indices of $G_1{\sim}G_2$ are given by:*

1. $M_1(G_1{\sim}G_2)=(r_2+1)+(r_1+1)+(n_1-1)(2r_1+1)+(n_2-1)(2r_2+1)+2r_1(m_1-r_1)+2r_2(m_2-r_2)$.
2. $M_2(G_1{\sim}G_2)=(r_2+1)(r_1+1)+r_1(r_1+1)(n_1-1)+r_2(r_2+1)(n_2-1)+(m_1-r_1)r^2+(m_2-r_2)r^2$.

$$3. M_2^m(G_1{\sim}G_2) = \frac{1}{(r_2+1)(r_1+1)} + \frac{n_1-1}{r_1(r_1+1)} + \frac{n_2-1}{r_2(r_2+1)} + \frac{m_1-r_1}{r_1^2} + \frac{m_2-r_2}{r_2^2}.$$

$$4. S_D(G_1{\sim}G_2) = \frac{r_2+1}{r_1+1} + \frac{r_1+1}{r_2+1} + \frac{r_1(n_1-1)}{r_1+1} + \frac{(r_1+1)(n_1-1)}{r_1} + \frac{r_2(n_2-1)}{r_2+1} + \frac{(r_2+1)(n_2-1)}{r_2} + 2(m_1-r_1) + 2(m_2-r_2).$$

$$5. H(G_1{\sim}G_2) = \frac{2}{r_1+r_2+2} + \frac{2(n_1-1)}{2r_1+1} + \frac{2(n_2-1)}{2r_2+1} + \frac{m_1-r_1}{r_1} + \frac{m_2-r_2}{r_2}.$$

$$6. I_n(G_1{\sim}G_2) = \frac{(r_1+1)(r_2+1)}{r_1+r_2+2} + \frac{r_1(r_1+1)(n_1-1)}{2r_1+1} + \frac{r_2(r_2+1)(n_2-1)}{2r_2+1} + \frac{r_1(m_1-r_1)}{2} + \frac{r_2(m_2-r_2)}{2}.$$

$$7. \chi_\alpha(G_1{\sim}G_2) = (r_1+r_2+2)^\alpha + (n_1-1)(2r_1+1)^\alpha + (n_2-1)(2r_2+1)^\alpha + 2^\alpha r_1^\alpha(m_1-r_1) + 2^\alpha r_2(m_2-r_2)^\alpha.$$

$$8. M_1^\alpha(G_1{\sim}G_2) = (r_2+1)^{\alpha-1} + (r_1+1)^{\alpha-1} + (n_1-1)(r_1^{\alpha-1} + (r_1+1)^{\alpha-1}) + (n_2-1)(r_2^{\alpha-1} + (r_2+1)^{\alpha-1}) + 2(m_1-r_1)r_1^{\alpha-1} + 2(m_2-r_2)r_2^{\alpha-1}.$$

Conclusion

In this paper, we obtained degree-based topological indices for different graph operations including join, corona product, strong product, tensor product, splice, and link of regular graphs. First, we computed M-polynomial of the aforesaid graph operations and later recovered many degree-based topological indices applying it.

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Conflict of Interest

The authors declare that there is no conflict of interests regarding the publication of this manuscript.

Orcid:

Raad Sehen Haoer:

https://orcid.org/0000-0001-5396-7598

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