Received: 19 February 2021 | Revised: 01 March 2021 | Accepted: 09 March 2021
DOI: 10.22034/ecc.2021.274069.1138
Eurasian Chemical Communications SAMI

FULL PAPER

On the ψK-polynomial of graph

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In this paper, for every k-degree distance we define ψ_k-Polynomial of a connected graph $G=(V,E)$ as $\psi_k(G,x,y)=\sum_{\delta_k\le i\le j\le\Delta_k}\psi_k(i,j)x^iy^j$, where $\psi_k(i,j)$ is the number of edges uv in G such that $\{d_k(v), d_k(u)\}=\{i,j\}$ and $d_k(v)$, $d_k(u)$ are the k^{th} degree distance of v and u, and δ_k, Δ_k are the minimum and maximum k^{th} distance degrees respectively. We compute the ψ2-Polynomial of some standard graphs and some graph operations. Also ψ2-Polynomial for honeycomb network and Graphene are obtained with their plotting in 3D.

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KEYWORDS

Second degree of a vertex; polynomial of graph; ψ2-polynomial; ψ_k-polynomial.

Introduction

Chemical Numerous graph polynomials are available in the recent literature and several among them are applicable in mathematical chemistry. For instance, the Hosoya polynomial [35]. A graph $G=(V,E)$ of order $n=|V(G)|$ and size $m=|E(G)|$ is connected if there exists a path between any pair of vertices in G. A network is a simple connected graph. If two vertices u and v of the graph G are adjacent, then the edge connecting them will be denoted by uv [34]. If $u,v\in V(G)$, then the distance $d_G(u,v)$ between u and v is defined as the length of a shortest path in G connecting them. In the graph G, the first degree of a vertex v, denoted $d(v)$, is the number of first neighbours (the number of edges incident with v). The maximum and minimum degrees among the vertices of G, are denoted by $\Delta=\Delta(G)$ and $\delta=\delta(G)$, respectively. Among other algebraic polynomials, the M-polynomial [9, 10] introducedin 2015, plays the same role in determining the closed-form of many degree-based topological indices [26-30, 33]. The main advantage of the ψ_k-polynomial is the wealth of information that it contains about degree-based graph invariants [10-23]. In this paper, we introduce the ψ_k-polynomial and, we find the ψ_k-polynomial for k=2 for some family of graphs, and Mycielskian of a graph, and corona product of C_n with $\bar{K}_n$. Also the ψ_k-polynomial of the honeycomb network graph and Graphene we are worked out with their 3-D plotting.

The ψ_k-Polynomial of graphs

In this section, we present the definition of ψ_k-polynomial of graphs, and establish the formulas of the exact values of ψ_k-Polynomial for some well-known graph classes with k=2.

Definition 2.1. *Let G be a graph the k^{th} degree of the vertex v is denoted by $d_k(v)$,and define as $d_k(v)=|\{u\in V(G):d(u,v)=k\}|$, where $d(u,v)$ is the distance between the vertices u and v. The minimum and maximum k^{th} degree of the graph G denoted by $\delta_k(G)$ and $\Delta_k(G)$ respectively, then for graph G let $\psi_k(i,j)(G)$ be the number of edge, $uv\in E(G)$ such that $\{d_k(u), d_k(v)\}=\{i,j\}$, The ψ_k-polynomial ofG is define as:*

$$\psi_k(G,x,y)=\sum_{\delta_k\le i\le j\le \Delta_k}\psi_k(i,j)(G)x^i y^i.$$

Theorem 2.1. *Let $G \cong P_n$ be a path with $n \geq 4$ vertices. Then*

$$\psi_2(G,x,y)=\begin{cases}3xy, & if\, n=4;\\ 2xy+2xy^2+(n-5)x^2y^2, & fn\ge 5.\end{cases}$$

Proof. **Case 1.** If n=4, then for every vertex $v \in V(P_4)$, the $d_2(v)$=1, then

$$\psi_2(G,x,y)=\sum_{\delta_2\le i,j\le \Delta_2}\psi_2(i,j)x^i y^j=3xy.$$

Case 2. If $n \geq 5$, then let $\{v_1, v_2,...v_n\}$ be vertices in G, then $d_2(v_i)$= 1, where i=1, 2, n-1, n, and $d_2(v_i)$=2, where i=3, 4,..n-2. Therefore

$$\psi_2(G,x,y)=\sum_{\delta_2\le i,j\le \Delta_2}\psi_2(i,j)x^i y^j$$

$$=2xy+2xy^2+(n-5)x^2y^2.$$

Theorem 2.2. *Let $G \cong C_n$ be a cycle graph with $n \geq 4$ vertices. Then*

$$\psi_2(G,x,y)=\begin{cases}4xy, & if\quad n=4;\\ nx^2y^2, & if\quad n\ge 5.\end{cases}$$

Proof. **Case 1.** If n= 4, then for every vertex $v_j \in G$ the $d_2(v_j)$= 1, where j= 1, 2, 3, 4. Therefore

$$\psi_2(G,x,y)=\sum_{\delta_2\le i,j\le \Delta_2}\psi_2(i,j)x^i y^j=4xy.$$

Case 2. If $n \geq 5$, then $d_2(v_j)$=2, for j= 1, 2,..n. Therefore

$$\psi_2(G,x,y)=\sum_{\delta_2\le i,j\le \Delta_2}\psi_2(i,j)x^i x^j=nx^2y^2.$$

Theorem 2.3. *Let $G \cong K_{s,r}$ be a complete bipartite graph of order $s+r$ with s, r> 1. Then $\psi_2(G,x,y)=(s+r)x^{s-1}y^{r-1}$.*

Proof. Let $X=\{u_1, u_2...u_s\}$ and $Y=\{v_1, v_2...v_r\}$ be a bipartition of $V(G)$. Let $u,v \in V(G)$, where $u \in X$ and $v \in Y$. Then If $s=r$, then $d_2(v_m)=d_2(u_n)=s-1$, where n=1, 2...s, m=1, 2...r. Therefore

$$\psi_2(G,x,y)=\sum_{\delta_2\le i,j\le \Delta_2}\psi_2(i,j)x^i y^j=srx^{s-1}y^{s-1}.$$

If $s \neq r$, then $d_2(v_n)=s-1$, $d_2(u_m)=r-1$. Therefore $\psi_2(G,x,y)=nmx^{s-1}y^{r-1}$.

Theorem 2.4. *Let $G \cong W_n$ be a wheel graph with $n+1$ vertices for $n \geq 4$. Then $\psi_2(G,x,y)=nx^{n-3}y^{n-3}+ny^{n-3}$.*

Proof. Let u is the center vertex of graph G, then $d_2(u)$=0, for n vertices in wheel graph $d_2(v_s)=n-3$, where s=1, 2...n. Therefore

$$\psi_2(G,x,y)=\sum_{\delta_2\le i,j\le \Delta_2}\psi_2(i,j)x^i y^j$$

$$=nx^{n-3}y^{n-3}+ny^{n-3}.$$

Theorem 2.5. *Let $G \cong F_p$ be a friendship graph with $n=2p+1$ vertices, and $p \geq 2$. Then*

$$\psi_2(G,x,y)=(\frac{n-1}{2})x^{n-3}y^{n-3}+(n-1)y^{n-3}.$$

Proof. Let $G \cong F_p$ friendship graph, and let u be the center vertices of G graph, then $d_2(u)$=0, and let $v=\{v_1, v_2,...v_s\}$, where $v \in V(G)$, then $d_2(v_s)=n-3$, where s=1, 2, 3...n. Therefore

$$\psi_2(G,x,y)=\sum_{\delta_2\le i,j\le \Delta_2}\psi_2(i,j)x^i y^j$$

$$=(\frac{n-1}{2})x^{n-3}y^{n-3}+(n-1)y^{n-3}.$$

Definition 2.2. *A firefly graph $F_{r,s,t}$ is a graph of order n vertices where $n= 2r+1+s+2t$ that consists of r triangles, s pendant edges and t pendant paths of length 2, all of them sharing a common vertex u.* [5].

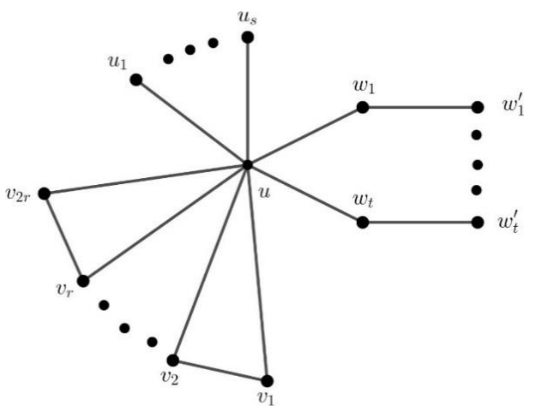

FIGURE 1 Firefly graph

Theorem 2.6. *Let $G \cong F_{r,s,t}$ be a firefly graph of order n with $r, s, t \geq 1$, and $\lambda=2r+s+t$. Then $\psi_2(G,x,y)=2rx^ty^{\lambda-2}+rx^{\lambda-2}y^{\lambda-2}+(s+t)x^ty^{\lambda-1}+txy^{\lambda-1}$.*

Proof. Let $G \cong F_{r,s,t}$ labeling as in Figure 1, then $d_2(u)=t$, and $\lambda=2r+s+t$, then $d_2(v_i)=\lambda-2$, where i=1, 2...2r, and $d_2(w_j)=\lambda$-1, where j=1, 2, 3,...t, and $d_2(\grave{w}_j)$=1, where j=1, 2, 3...t. and $d_2(u_m)=\lambda$-1, where m=1, 2, 3...s. Therefore

$$\psi_2(G,x,y) = \sum_{\delta_2 \le i,j \le \Delta_2} \psi_2(i,j)x^i y^j$$
$$= 2rx^t y^{\lambda-2} + rx^{\lambda-2}y^{\lambda-2} + (s+t)x^t y^{\lambda-1} + txy^{\lambda-1}.$$

Theorem 2.7. *Let $G_1 \cong C_n$ is cycle graph, with n vertices, and $G_2 \cong \bar{K}_m$ is complete graph with m vertices. Then*

$$\psi_2(G_1 \circ \overline{G_2}, x, y) = nmx^{m+1}y^{2(m+1)} + nx^{2(m+1)}y^{2(m+1)}.$$

Proof. let G_1, G_2, be a graphs with n, m vertices as in figure 2. then its easy to see that $d_2(v_r)=2m+2$, where r=1, 2, 3...m, and $d_2(u_s)=m+1$, where s=1, 2, 3...m. Therefore

$$\psi_2(G_1 \circ \overline{G_2}, x, y) = \sum_{\delta_2 \le i,j \le \Delta_2} \psi_2(i,j)x^i y^j$$
$$= nx^{2(m+1)}y^{2(m+1)} + nmx^{(m+1)}y^{2(m+1)}.$$

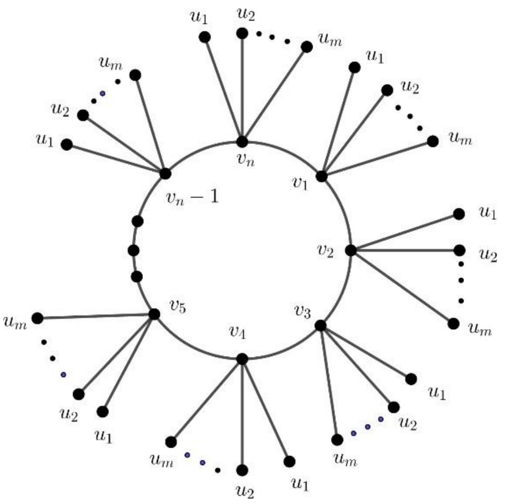

FIGURE 2 ($C_n \circ \bar{K}_m$)

Theorem 2.8. *Let G a connected graph with n≥2 vertices. Then $\psi_2(G,x,y) = \frac{1}{2}n(n-1)$ if $G \cong K_n$*

Proof. Let $G \cong K_n$, and let $V=\{v_1, v_2,..v_n\}$, where $v \in V(G)$, then $d_2(v_i)=0$, for i=1, 2,..n. Therefore

$$\psi_2(G,x,y) = \sum_{\delta_2 \le i,j \le \Delta_2} \psi_2(i,j)x^i y^j = \frac{n(n-1)}{2}.$$

In the other direction if G a connected graph with n≥2 vertices, and $\psi_2(G,x,y) = \frac{1}{2}n(n-1)$ so for every $v \in G$ $d_2(v)=0$, then $G \cong K_n$.

Theorem 2.9. *Let G a connected graph with n≥2 vertices. Then $\psi_2(G,x,y)=cy^r$ if $G \cong S_n$. where $c=(n-1)$, and $r=n-2$.*

Proof. Let G a connected graph with n≥2, and $G \cong S_n$. Then $d_2(u)=0$, where u is the center vertex in S_n, and $d_2(v_i)=n-2$, where i=1, 2,..n-1, then

$$\psi_2(G,x,y) = \sum_{\delta_2 \le i,j \le \Delta_2} \psi_2(i,j)x^i y^j = (n-1)y^{n-2}.$$

ψ_2-Polynomial of Mycielskian of a Graph

Definition 3.1. *Let G a graph with $V(G)=\{v_1, v_2, v_3,..., v_n\}$. The Mycielski graph or Mycielskian $\mu(G)$ of a graph G is obtained by edges from each u_i from the new set of vertices $U=\{u_1, u_2, u_3,..., u_n\}$ to the vertices v_j, if the corresponding vertex u_i is adjacent to the v_j in G, and add another new vertex u, and add edges to all elements in vertex u* [25].

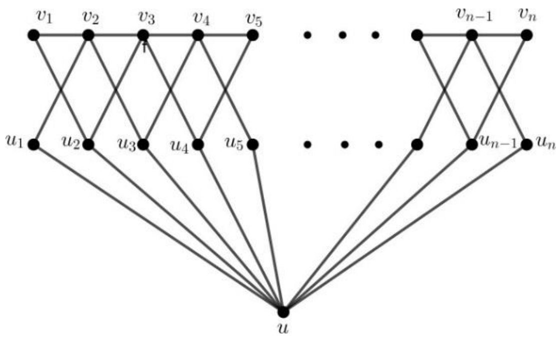

FIGURE 3 Mycielskian graph of P_n

Theorem 3.1. *Let $G \cong \mu(P_n)$, where P_n is path with n≥3 vertices. Then*

$$\psi_2(G,x,y) = \begin{cases} 5x^2y^2 \text{ if } n=2;\ x^3y^3 + 4x^3y^4 + 4x^2y^4, \text{ if } n=3; \\ 10x^5y^4 + 3x^4y^4, \text{ if } n=4, \\ 2x^4y^4 + 2x^4y^6 + 2x^4y^{n+1} + 2x^4y^{n+2} \\ +(n-5)x^6y^6 + \lambda, \text{ if } n \ge 5. \end{cases}$$

Where

$\lambda = 2x^6y^{n+1} + (2n-10)x^6y^{n+2} + 4x^ny^{n+1} + (n-4)x^ny^{n+2}$.

Proof. Let $G \cong \mu(P_n)$ as in Figure 3. then we have

__Case 1.__ If n=2, then $\mu(G) \cong C_5$, then by

Theorem 2.2 we have the result.

Case 2. If n=3, then $\psi_2(u)=d_2(u_2)=3$, $\psi_2(v_2)=2, d_2(v_1)=d_2(v_3)=4$, $d_2(u_1)=d_2(u_2)=4$. Therefore $\psi_2(G,x,y)=\sum_{\delta_2\le i,j\le\Delta_2}\psi_2(i,j)x^iy^j$

$$=x^3y^3+4x^3y^4+4x^2y^4.$$

Case 3. If n=4, then we get the $d_2(v_i)=4$, where i=1, 2, 3.., n, and $d_2(u_j)$=5, where j=1, 2, 3..., n, and $d_2(u)$=3. Therefore

$$\psi_2(G,x,y)=\sum_{\delta_2\le i,j\le\Delta_2}\psi_2(i,j)x^iy^j=10x^5y^4+3x^4y^4.$$

Case 4. If $n\geq5$, then $d_2(v_i)$=4, where i=1, 2, v_{n-1}, v_n, and $d_2(v_i)$=6, where i=3, 4,...v_{n-3}, and $d_2(u)=n$, and $d_2(u_j)=n+1$, where j=1, 2, n-1, n, and $d_2(u_j)=n+2$, where j=3, 4,...n-3. Therefore

$$\psi_2(G,x,y)=\sum_{\delta_2\le i,j\le\Delta_2}\psi_2(i,j)x^iy^j$$

$=2x^4y^4+2x^4y^6+2x^4y^{n+1}+2x^4y^{n+2}+(n-5)x^6y^6+\lambda$.

where

$\lambda=2x^6y^{n+1}+(2n-10)x^6y^{n+2}+4x^ny^{n+1}+(n-4)x^ny^{n+2}$.■

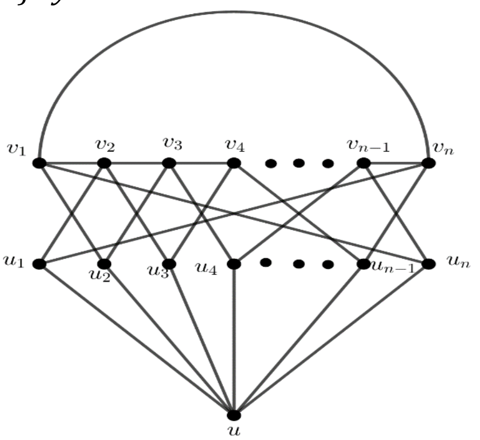

FIGURE 4 Mycielskian graph of C_n

Theorem 3.2. *Let $G\cong\mu(C_n)$, where C_n be a cycle with $n\geq3$ vertices. Then*

$$\psi_2(G,x,y)=\begin{cases}3x^3y^4+9x^4y^4, if n=3;\\ 4x^4y^4+12x^4y^5, if n=4;\\ nx^6y^6+n(x^n+2x^6)y^{n+2}, if n>4.\end{cases}$$

Proof. Let G Mycielskian graph of C_n as in Figure 4, Then we have 3 case

Case 1. If n=3, then $d_2(v_i)=d_2(u_j)=4$, where i=1, 2,..., n, and j=1, 2,..., n and $d_2(u)$=3. Therefore

$$\psi_2(G,x,y)=\sum_{\delta_2\le i,j\le\Delta_2}\psi_2(i,j)x^iy^j$$

$$=3x^3y^4+9x^4y^4.$$

Case 2. If n=4, then by the Figure 4. $d_2(v_i)=d_2(u)=4$, and $d_2(u_j)=5$, where i=1, 2,...n and j=1, 2,...n. Therefore

$$\psi_2(G,x,y)=\sum_{\delta_2\le i,j\le\Delta_2}\psi_2(i,j)x^iy^j$$

$=4x^4y^4+12x^4y^5$.

Case 3. If $n>4$, then $d_2(u)=n$, $d_2(v_i)=6$, and $d_2(u_j)=n+2$, where i=1, 2,...n, and j=1, 2...n. Therefore

$$\psi_2(G,x,y)=\sum_{\delta_2\le i,j\le\Delta_2}\psi_2(i,j)x^iy^j$$

$=nx^6y^6+n(x^n+2x^6)y^{n+2}$.

Theorem 3.3. *Let $G\cong\mu(K_n)$, with $n\geq3$ vertices. Then*

$$\psi_2(G,x,y)=\begin{cases}3x^3y^4+9x^4y^4, if n=3;\\ nx^ny^{2n-2}+(n^2+\sum_{i=2}^{n-2}(n-i))x^{2n-2}y^{2n-2}, if n\geq4.\end{cases}$$

Proof. Let G be Mycielskian graph of complete graph K_n with $n\geq3$ vertices, then we have two cases.

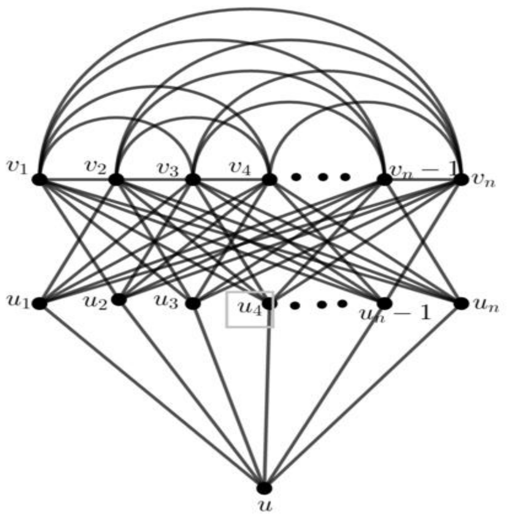

FIGURE 5 Mycielskian graph of K_n

Case 1. If n=3, then $\mu(G)\cong\mu(C_3)$, then we have the result by Theorem 3.2.

Case 2. If $n \geq 4$ then by the Figure 5. we $d_2(v_i)=d_2(u_j)=2n-2$, and $d_2(u)=n$, and $\psi_2(2n-2,2n-2)=(n^2+\sum_{i=2}^{n-2}(n-i))$. Therefore

$$\psi_2(G,x,y)=\sum_{\delta_2\leq i,j\leq\Delta_2}\psi_2(i,j)x^iy^j$$

$$=nx^ny^{2n-2}+(n^2+\sum_{i=2}^{n-2}(n-i))x^{2n-2}y^{2n-2}.$$ ■

ψ_2-Polynomial of A honeycomb network and Graphene

Definition 4.1. *A honeycomb network HN (n) can be built from a hexagon in various ways* [31, 33]. *The honeycomb network HN (1) is a hexagon. The honeycomb network HN (2) is obtained by adding six hexagons to the boundary edges, HN (n) is obtained from HN (n-1) by adding a layer of hexagons around the boundary of HN (n-1). The number of vertices of HN (n) is $6n^2$ and the number of edges of a honeycomb graph is $9n^2$-3n* [8].

TABLE 1 The number of edges of honeycomb network with $n \geq 2$

$\psi_2(i,j)$	$\psi_2(3,3)$	$\psi_2(3,4)$	$\psi_2(4,4)$	$\psi_2(4,6)$	$\psi_2(6,6)$
no.edge	6	12	$12(n-2)$	$6(n-1)$	$9n^2-21n+12$

Theorem 4.1. *Let $G \cong HN$ (n) is a honeycomb network with n hexagon with $n \geq 1$. Then*

$$\psi_2(G,x,y)=\begin{cases}6xy, & if n=1;\\ 6x^3y^3+12x^3y^4+(12n-24)x^4y^4 & \\ +6(n-1)x^4y^6+\alpha, if n\geq 2 & \end{cases}$$

where $\alpha=(9n^2-21n+12)x^6y^6$.

Proof. **Case 1.** If $n=1$, then $G \cong C_6$, then by Theorem 2.2. $\psi_2(G,x,y)=6xy$.

Case 2. If $n \geq 2$ in the honeycomb network *HN* (*n*), the *HN* (2) obtain by add six hexagon to the boundary edge in *HN* (1), also in *HN* (3) obtain by add 12 hexagonto the boundary edges in *HN* (2), so we get *HN* (*n*) obtain by add $6(n-1)$ hexagon to the boundary edge in *HN* (n-1), then the number of edge of *HN* (n-1) is $9(n-1)^2-3(n-1)$, then we get $d_2(v)=6$ for all $v \in HN$ (n-1) with *HN* (n) graph, then $\psi_2(6,6)=9(n-1)^2-3(n-1)$, then as in the Table 1., we get

$$\psi_2(G,x,y)=\sum_{\delta\leq i,j\leq\Delta_2}\psi_2(i,j)x^iy^j$$

$=6x^3y^3+12x^3y^4+12(n-2)x^4y^4+6(n-1)x^4y^6+\alpha$. where $\alpha=(9n^2-21n+12)x^6y^6$.■

Graphene: is an atomic-scale honeycomb lattice made of the carbon atoms. It is the first 2*D* material that was isolated from graphite in 2004 by Professor Andre Geim and Professor Kostya Novoselov. Graphene is 200 times stronger than steel, one million times thinner than a human hair, and the world's most conductive material. So it has captured the attention of scientists, researchers, and industries worldwide. It is one of the most promising nanometres because of its unique combination of superb properties, which opens a way for its exploitation in a wide spectrum of applications ranging from electronics to optics, sensors, and biodevices. Also, it is the most effective material for electromagnetic interference (EMI) shielding. Sridhara was calculated some topological indices of graphene in [30]. The structure of graphene $G(s,t)$ with t rows and s columns in Figure 3.

Proposition 4.2. *Let $G \cong G_{s,t}$ be a graph of Graphene with $s=t=1$. Then $\psi_2(G,x,y)=6x^2y^2$.*

Proof. Let G a graph, if $s=r=1$, then $G \cong C_6$. Therefore by Theorem 2.2 we have the result.■

Table 2 The number of edges of Graphene with $t=1$, and $s>1$

R	$\psi_2(2,2)$	$\psi_2(2,3)$	$\psi_2(3,4)$	$\psi_2(4,4)$
1	2	4	4	$(5s-9)$

Proposition 4.3. *Let $G \cong G_{s,t}$ be a graph of Graphene with t=1, s>1. Then* $\psi_2(G,x,y)=2x^2y^2+4x^2y^3+4x^3y^4+(5s-9)x^4y^4$.

Proof. For a graph G, if t=1, s>1, then we get the first and last columns are similar, and the $d_2(v)$ for all the vertices v in the hexagons between the first and the last column have its equal, then from the Table 2. we get $\psi_2(2, 2)$=2, $\psi_2(2, 3)$=4, $\psi_2(3, 4)$=4, $\psi_2(4, 4)$=5s-9. Therefore

$$\psi_2(G,x,y)=\sum_{\delta_2\le i\le j\le\Delta_2}\psi_2(i,j)(G)x^iy^i$$

$=2x^2y^2+4x^2y^3+4x^3y^4+(5s-9)x^4y^4$.■

TABLE 3 The number of edges of Graphene with t>2, and s=1

column	$\psi_2(2,2)$	$\psi_2(2,3)$	$\psi_2(3,3)$	$\psi_2(3,4)$	$\psi_2(3,5)$	$\psi_2(4,5)$	$\psi_2(5,5)$
1	2	4	t-2	4	2t-4	2	2t-5

TABLE 4 The number of edges of Graphene with t>1, and s>1

Row	$\psi_2(2,3)$	$\psi_2(3,3)$	$\psi_2(3,4)$	$\psi_2(3,5)$	$\psi_2(4,4)$	$\psi_2(4,6)$	$\psi_2(5,5)$	$\psi_2(5,6)$	$\psi_2(6,6)$
1	2	1	3	3	1	2(s-2)	0	s	2s-3
2	0	1	1	1	1	0	1	0	3s-4
3	0	1	0	0	2	0	1	0	3s-4
4	0	1	0	0	2	0	1	0	3s-4
5	0	1	0	0	2	0	1	0	3s-4
6	0	1	0	0	2	0	1	0	3s-4
.	.	.	.	.	.	.	.	.	.
.	.	.	.	.	.	.	.	.	.
t-1	0	1	1	0	1	0	1	1	3s-4
t	2	1	1	3	1	2(s-2)	0	s-1	0
Total	4	t	8	2t-4	4s-8	2s	t-2	2t-4	3st-4t-4s+5

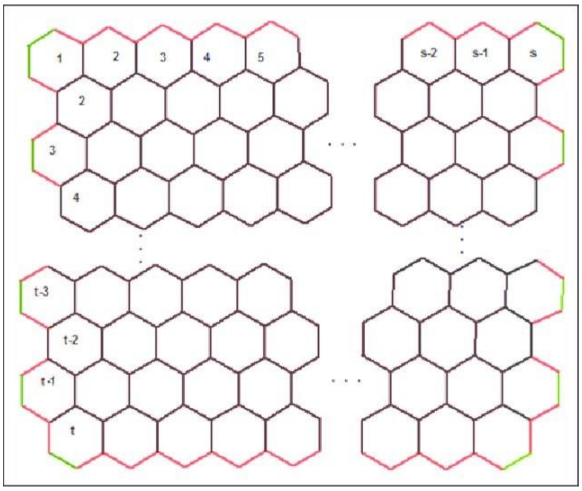

FIGURE 6 The structure of graphene $G(s, t)$

Proposition 4.4. *Let $G \cong G_{s,t}$ be graph of Graphene with t>2, s=1. Then*

$$\psi_2(G,x,y)=\begin{cases}2x^2y^2+4x^2y^3+4x^3y^4+x^4y^2, if\ t=2;\\ 2x^2y^2+4x^2y^3+(t-2)x^3y^3\\ +4x^3y^4+(2t-4)x^3y^5+\beta; if\ t>2.\end{cases}$$

where $\beta=2x^4y^5+(2t-5)x^5y^5$.

Proof. **Case 1.** let G be graph of Graphene, if s=1, and t=2, then we have two hexagons have only one edge and two vertices are common, and the two hexagons its similar, then $\psi_2(2, 2)$=2, $\psi_2(2, 3)$=4, $\psi_2(3, 4)$=4, $\psi_2(4, 4)$=1. Therefore

$$\psi_2(G,x,y)=\sum_{\delta_2\le i\le j\le\Delta_2}\psi_2(i,j)(G)x^iy^i$$

$=2x^2y^2+4x^2y^3+4x^3y^4+x^4y^4$.

Case 2. Let G be graph of Graphene with s=1, and t>2, then the column t_1 similar with t_n, and the t_2 similar with t-1, and t_3 to t-3 are similar then from the Table.3 we get

$$\psi_2(G,x,y)=\sum_{\delta_2\le i\le j\le\Delta_2}\psi_2(i,j)(G)x^iy^i$$

$=2x^2y^2+4x^2y^3+(t-2)x^3y^3+4x^3y^4$
$+(2t-4)x^3y^5+\beta$.

where $\beta = 2x^4y^5 + (2t-5)x^5y^5$.

Theorem 4.5. *Let $G \cong G_{s,t}$ be a graph of Graphene with $t>1$, $s>1$. Then*

$\psi_2(G,x,y) = 4x^2y^3 + tx^3y^3 + 8x^3y^4 + (2t-4)x^3y^5 + (4s-8)x^4y^4 + \gamma.$

where $\gamma = 2sx^4y^6 + (2t-4)x^5y^6 + (t-2)x^5y^5 + (3st-4s-4t+5)x^6y^6$.

Proof. Let G is Graphene with t row and s column, we compute the $d_2(v)$ of all vertices $v \in G$, and compute the number of edge with initial point $d_2(v)$ and endpint $d_2(u)$ of the edge $uv \in G$, then from the Table 4. we get $\psi_2(2,3)=4$, and $\psi_2(3,3)=t$, $\psi_2(3,4)=8$, $\psi_2(3,5)=2t-4$, $\psi_2(4,4)=2s$, $\psi_2(4,6)=t-2$, $\psi_2(5,5)=2t-4$, $\psi_2(6,6)=3st-4(s-t)+s$.

Therefore $\psi_2(G,x,y) = 4x^2y^3 + tx^3y^3 + 8x^3y^4 + (2t-4)x^3y^5 + (4s-8)x^4y^4 + \gamma$.

where $\gamma = 2sx^4y^6 + (2t-4)x^5y^6 + (t-2)x^5y^5 + (3st-4s-4t+5)x^6y^6$.■

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How to cite this article: Ammar Alsinai, Anwar Alwardi, Mohammad Reza Farahani*, Soner Nandappa D. On the ψk-polynomial of graph. *Eurasian Chemical Communications*, 2021, 3(4), 219-226. **Link:** http://www.echemcom.com/article_128248.html