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FULL PAPER

Eccentricity version F-index of some graph operations

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The eccentricity version F-index of a graph G is defined as the summation of cube of all the vertex eccentricities of G. In this report, the eccentricity version of F-index of the generalized hierarchical product of graphs was computed. Further, we obtained some explicit expressions of eccentricity version of F-index for S-sum graphs, Cartesian product graphs and cluster product graphs. Hence applying those outcomes, we arrived at the eccentricity version of F-index for Chemical graphs and nanostructures.

KEYWORDS

Eccentricity; topological index; F-index; product graphs.

Introduction

Eccentricity version of F-index of a graph G is given by

$$F(G) = \sum_{v \in V(G)} \varepsilon_G(v)^3.$$

and E(G) be the vertex and edge sets of a graph G, respectively. For any two nodes x;y∈V(G), the distance between them, denoted by $d_G(x;y)$ is defined by the number of edges in the shortest path connecting x and y. The eccentricity of a vertex v∈V(G), denoted by $\varepsilon_G(v)$, is defined as the largest distance from u to any other vertex of G. In the field of mathematics and chemistry, the topological indexes play an important role to model different properties and activities of Chemical structures and networks. A topological index is a mapping from the collection of all graphs to the set of real numbers that remains invariant under graph isomorphism. Utilization of such indices in Chemistry and biology started in 1947 when chemist H. Wiener [1] presented the wiener index for searching boiling points of alkane. Now, hundreds of topological indices have been defined in Chemical literatures with various applications and many mathematics properties. The present work deals with vertex eccentricity based topological indices. In 1997, Sharma, et al. [2] introduced the eccentric connectivity index of molecular graph. In [3] Fatha-likhani et al. studied total eccentricity of some graph operations. Some studies on average eccentricities are also found in literature [5,4].

Barriere et al. introduced the hierarchical product graph and also reported a generalization of both Cartesian and the hierarchical product of graphs, namely the generalized hierarchical product of graphs in 2009, [6, 7]. Some special cases of generalized hierarchical product are the Cartesian product, F-sum product and cluster product.

In this paper, the eccentricity version of F-index of the generalized hierarchical product graph and some special cases of the eccentricity version of F-index of the Cartesian product, S-sum and cluster product graphs are computed.

Eccentricity version of F-index of product graphs

In this section, we define generalized hierarchical product of two graphs G and H and therefore we can calculate the eccentricity version of F-index of the generalized hierarchical product of graphs G and H.

Definition 1. [6] Let *G &H* be two connected graphs and $\phi \neq U \subseteq V(G)$. Then the vertex setoff the generalized hierarchical product *G(U)⊓H* is *V(G)×V(H)* and two vertices $(u_r;v_i)$ and $(u_s;v_k)$ are adjacent if and only if *[$u_r=u_s \in U$* and *$v_iv_k \in E(H)$]* or *[$v_i=v_k \in V(H)$* and *$u_ru_s \in E(G)$].* Since $\phi \neq U \subseteq V(G)$ the distance through U between x and y is denoted by $d_{G(u)}(x;y)$ and if one of the vertex x and y belongs to U, then *$d_{G(u)}(x;y)=d_G(x;y)$.*

Now, we represent some necessary invariants related to U in G as follows:

$$\varepsilon_{G(U)}(v) = max_{u \in V(G)} d_{G(U)}(u,v)$$

$$\zeta(G(U)) = \sum_{v \in V(G)} \varepsilon_{G(U)}(v)$$

$$E_1(G(U)) = \sum_{v \in V(G)} \varepsilon_{G(U)}(v)^2$$

$$F(G(U)) = \sum_{v \in V(G)} \varepsilon_{G(U)}(v)^3.$$

Lemma 1. [7] Let *G & H* be two connected graphs then $\varepsilon_{G(U)\sqcap H}(u,v) = \varepsilon_{G(U)}(u) + \varepsilon_H(v),$ that $\phi \neq U \subseteq V(G).$

Theorem 1. Let *G & H* be two connected graphs and $\phi \neq U \subseteq V(G),$ then

$$F_\varepsilon(G(U)\sqcap H) = |V(H)|F_\varepsilon(G(U)) + |V(G)|F_\varepsilon(H)$$
$$+3E_1(G(U))\zeta(H) + 3E_1(H)\zeta(G(U)).$$

Proof: Let *$u_r \in V(G)$ & $v_l \in V(H)$.* Then by using definition of eccentricity version of F-index and lemma 1, we have

$$F_\varepsilon(G(U)\sqcap H) = \sum_{(u_r,v_l)\in V(G(U)\sqcap H)} \varepsilon_{G(U)\sqcap H}(u_r,v_l)^3$$
$$= \sum_{u_r \in V(G)} \sum_{v_l \in V(H)} (\varepsilon_{G(U)}(u_r) + \varepsilon_H(v_l))^3 = |V(H)| \sum_{u_r \in V(G)} \varepsilon_{G(U)}(u_r)^3 + |V(G)| \sum_{v_l \in V(H)} \varepsilon_H(v_l)^3$$
$$+3 \sum_{u_r \in V(G)} \varepsilon_{G(U)}(u_r)^2 \sum_{v_l \in V(H)} \varepsilon_H(v_l) + 3 \sum_{u_r \in V(G)} \varepsilon_{G(U)}(u_r) \sum_{v_l \in V(H)} \varepsilon_H(v_l)^2$$
$$= |V(H)|F_\varepsilon(G(U)) + |V(G)|F_\varepsilon(H) + 3E_1(G(U))\zeta(H) + 3E_1(H)\zeta(G(U)).$$

Hence the desired result follows.

As another very important graph operation, the Cartesian product of graphs is defined as follows:

Definition 2. Let *G & H* be two connected graphs. The Cartesian product G H with vertex set *V(G)×V(H)* and vertices $(u_r;v_i)$ and $(u_s;v_k)$ are adjacent if and only if *[$u_r=u_s \in V(G)$ and $v_iv_k \in E(H)$]* or *[$v_i=v_k \in V(H)$ and $u_ru_s \in E(G)$],* where *r;s=1;2;...;|V(G)|* and *i;k=1;2;...;|V(H)|.* It is clear that, if U=V(G), then $G(U)\sqcap H \cong G \otimes H$. Hence by using theorem 1, considering U=V(G), we get the following result.

Corollary 1. Let *G & H* be two connected graphs. Then

$$F_\varepsilon(G \otimes H) = |V(H)|F_\varepsilon(G) + |V(G)|F_\varepsilon(H)$$
$$+3E_1(G)\zeta(H) + 3E_1(H)\zeta(G).$$

Eccentricity version of F-index of S-sum graphs

Let G be a connected graph with n vertices. The subdivision graph S(G) of G is the graph obtained by inserting an additional vertex to each edge of G, that is, each edge of G is replaced by a path of length two. Also, the vertices of a line graph L(G) are the edges of G and two edges of G that share a vertex are considered to be adjacent in L(G).

Definition 3. [8] Let *G & H* be two connected graphs, the S-sum $G+_SH$ of G and H is a graph with vertex set *(V(G)×E(G))×V(H)* and vertices *$(u_r;v_i)$* and *$(u_s;v_k)$* are adjacent if and only if *[$u_r=u_s \in V(G)$ and $v_iv_k \in E(H)$]* or *[$v_i=v_k \in V(H)$ and $u_ru_s \in E(S(G))$].* If $\phi \neq U = V(G) \subset V(S(G)),$ then $G+_S H \cong S(G)(U)\sqcap H \cong S(G)(V(G))\sqcap H.$

Lemma 2. [8] Let $G=T_n(n\geq 2)$ be a tree with n vertices and if U=V(G), then

(i) for each vertex v∈U, we have $\varepsilon_{S(G)(U)}(v) = 2\varepsilon_G(v),$

(ii) for each vertex *v∈V(S(G))-U,* we have $\varepsilon_{S(G)(U)}(v) = 2\varepsilon_{L(G)}(v)+1.$

Now using theorem 1, we can compute the eccentricity version of F-index as follows:

Theorem 2. Let T_n (n≥2) be a tree with n vertices and H be an arbitrary connected graph and $\phi = U = V(T_n) \subset V(S(T_n))$. Then

$$\begin{aligned} F_\varepsilon(T_n +_S H) &= |V(H)|[8F_\varepsilon(T_n) + 8F_\varepsilon(L(T_n)) + 12E_1(L(T_n)) \\ &+ 6\zeta(L(T_n)) + n - 1] + (2n-1)F_\varepsilon(H) + 6[2\zeta(H)E_1(T_n) \\ &+ \zeta(T_n)E_1(H)] + 6[2\zeta(H)E_1(L(T_n)) + E_1(H)\zeta(L(T_n))] \\ &+ 12\zeta(H)\zeta(T_n) + 3(n-1)[\zeta(H) + E_1(H)]. \end{aligned}$$

Proof: To calculate $F_\varepsilon(T_n +_S H)$, first we find the value of $F_\varepsilon(S(T_n)(U))$ by using lemma2 as follows:

$$\begin{aligned} F_\varepsilon(S(T_n)(U)) &= \sum_{v \in V(S(T_n))} \varepsilon_{S(T_n)(U)}(v)^3 = 8\sum_{v \in V(T_n)} \varepsilon_{T_n}(v)^3 + \sum_{v \in V(L(T_n))} (2\varepsilon_{L(T_n)}(v) + 1)^3 \\ &= 8F_\varepsilon(T_n) + 8F_\varepsilon(L(T_n)) + 12E_1(L(T_n)) + 6\zeta(L(T_n)) + n - 1. \end{aligned}$$

Similarly, using lemma 2, we have

$([E_1(S(T_n)(U)) = 4E_1(T_n) + 4E_1(L(T_n)) + 4\zeta(L(T_n)) + n - 1)$

And $(\zeta(S(T_n)(U)) = 2\zeta(T_n) + 2\zeta(L(T_n)) + n - 1.$.

Combining these results in theorem 1, we get the result in theorem 2.■

Let C_n be a cycle graph with n (≥3) number of vertices and $U=V(C_n)$. Then $\varepsilon_{S(C_n)(U)}(v) = n$ for any vertex $v \in V(S(C_n))$. Then by using theorem 1, we can obtain the following theorem.

Theorem 3. For any integer number n (≥3), let C_n and H be two arbitrary cycle and connected graphs, and $\phi \neq U = V(C_n) \subset V(S(C_n))$. Then:

$$F_\varepsilon(C_n +_S H) = 2n[F_\varepsilon(H) + 3nE_1(H) + 3n^2\zeta(H) + n^3|V(H)|].$$

Proof: Using theorem 1, we get

$$\begin{aligned} F_\varepsilon(C_n +_S H) &= |V(H)|F_\varepsilon(S(C_n)(U)) + |V(S(C_n))|F_\varepsilon(H) \\ &+ 3E_1(S(C_n)(U))\zeta(H) + 3E_1(H)\zeta(S(C_n)(U)) \\ &= |V(H)|2n^4 + 2nF_\varepsilon(H) + 6n^3\zeta(H) + 6n^2E_1(H).+ \end{aligned}$$

Example 1. [11] Let *Γ=TUHC₆[2n;2]* be the Zigzag polyhex nanotube (see figure 1); then $\Gamma = C_n +_S P_2$. Thus using above theorem, we get $F_\varepsilon(\Gamma) = 4n(n+1)^3$.

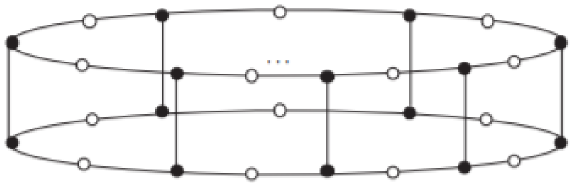

FIGURE 1 [11]: The zig-zay polyhex nanotube of $TUHC_6[2n;2]$

Eccentricity version F-index of cluster product of graphs

In this section, we calculated eccentricity version of F-index of cluster product of graphs. The cluster product of two graphs G and H are defined as follows:

Definition 4. [12] Let G{H} be the cluster product graph, that obtained by taking one copy of G and |(G)| copies of a rooted graph H by identifying the rooted of the i^{th} copy of H with the i^{th} vertex of G; *i=1;2;...;|V(G)|*.

Let, x is a root vertex of H. Now, if, $\phi \neq U = \{x\} \subset V(H)$ then $G\{H\} \cong H(U) \,´\, G \cong H(\{x\})$ G.

Also, let $d(x|G) = \sum_{v \in V(G)} d_G(x,v), d(x|G)^2 = \sum_{v \in V(G)} d_G(x,v)^2$ and $d(x|G)^3 = \sum_{v \in V(G)} d_G(x,v)^3,$ where $d_G(u;v)$ denotes the distance between vertices u and v of G. Hence, using theorem 1, we get the following result.

Theorem 4. Let *G & H* be two connected graphs and x is a root vertex of H. Then:

$$\begin{aligned} F_\varepsilon(G\{H\}) &= |V(H)|\varepsilon_H(x)^3 + 3\varepsilon_H(x)^2 d(x|H) + 3\varepsilon_H(x) \\ &+ d(x|H)^2 + d(x|H)^3] + |V(H)|F_\varepsilon(G) + 3\zeta(G)[|V(H)|\varepsilon_H(x)^2 \\ &+ 2\varepsilon_H(x)d(x|H) + d(x|H)^2] + 3E_1(G)(|V(H)|\varepsilon_H(x) + d(x|H)). \end{aligned}$$

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Proof: Suppose $\phi \neq U = \{x\} \subset V(H)$, then

$$\begin{aligned} F(H(U)) &= F(H\{x\})|V(H)|\varepsilon_H(x)^3 \\ &+ 3\varepsilon_H(x)^2 d(x|H) + 3\varepsilon_H(x)d(x|H)^2 + d(x|H)^3. \end{aligned}$$

Similarly

$$\begin{aligned} E_1(H(U)) &= E_1(H\{x\}) \\ &= |V(H)|\varepsilon_H(x)^2 + 2\varepsilon_H(x)d(x|H) + d(x|H)^2 \end{aligned}$$

And $\zeta(H(U)) = \zeta(H\{x\}) = |V(H)|\varepsilon_H(x) + d(x|H)$.

Now from theorem 1, we have

$$\begin{aligned} F_\varepsilon(H(U) \,´\, G) &= F_\varepsilon(G\{H\}) = |V(G)|F_\varepsilon(H(U)) + |V(H)|F_\varepsilon(G) \\ &+ 3E_1(H(U))\zeta(G) + 3E_1(G)\zeta(H(U)). \end{aligned}$$

Thus combining, (1), (2) and (3) in the above result, we can obtain the desired result.

Conclusion

In this work, we calculated the eccentricity version of F-index of the generalized hierarchical product of graphs. Further, we obtained some explicit expressions of eccentricity version of F-index for S-sum graphs, Cartesian product graphs and cluster product graphs. Also, using those results, we obtained the eccentricity version of F-index for chemical graphs and nanostructures. There are more related studies available in the literautre for future research and research [13-40].

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